HW #3

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1 Non-homogeneous polynomials and capacity bounds

We first define a non-homogeneous version of Lorentzian polynomials.

Definition 1.1. Given $p \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$, we say p is **completely/strongly log-concave** if for all $k \geq 0$ and all $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n_{\geq 0}$ we have that $D_{\boldsymbol{v}_1} \cdots D_{\boldsymbol{v}_k} p$ is log-concave (or identically zero) in the strict positive orthant.

Recall we define a linear operator N on polynomials via $N[\boldsymbol{x}^{\boldsymbol{\kappa}}] = \frac{x_1^{\kappa_1} \cdots x_n^{\kappa_n}}{\kappa_1! \cdots \kappa_n!}$.

Definition 1.2. Given $p \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$, we say p is **DL** if $N[p]$ is completely log-concave.

Note that by the previous homework, Lorentzian polynomials are completely/strongly log-concave, and DL polynomials in the original homogeneous sense are DL.

1.1 Exercises

1. Prove that given a matrix $A \in \mathbb{R}^{n \times m}_{\geq 0}$, if p is completely log-concave then so is $p(A\boldsymbol{x})$.

2. Let $p(\boldsymbol{x}) = \sum_{k=0}^{d} p_k(\boldsymbol{x})$ be such that p_k is k-homogeneous. Prove that p is completely log-concave if and only if
$$q(\boldsymbol{x}) = \sum_{k=0}^{d} \frac{y^{d-k}}{(d-k)!} \cdot p_k(\boldsymbol{x})$$
is completely log-concave. Note that this is equivalent to saying that p is DL if and only if its homogenization is DL.

3. Given an example of a completely log-concave polynomial such that its homogenization is not completely log-concave.

4. Finish the proof started in class that the operation $p(\boldsymbol{x}) \mapsto p(x_1, x_1, x_3, \ldots, x_n)$ preserves DL. (Note that by the above exercises, you may WLOG assume p is homogeneous DL.) Show that this implies that DL is preserved under products, even in the non-homogeneous case. (**Hint:** Recall the proof in the homogeneous case from the lecture.)

5. Using the previous exercises and results from the lecture, prove that completely log-concave polynomials are closed under taking products.

6. Finish the proof started in class that the homogeneous independent set generating polynomial of a matroid M,
$$q_M(\boldsymbol{x}, y) := \sum_{S \in \mathcal{I}(M)} \boldsymbol{x}^S y^{n-|S|},$$
is Lorentzian. Here, $\mathcal{I}(M)$ is the set of all independent sets of M and n is the size of the ground set of M (i.e., the number of x variables). (Note that we did not have to divide by factorials here when homogenizing. This is actually a bit of a mystery, since in general the factorials are required.)

7. Suppose $p(x,z) = axz + bx + cz + d \in \mathbb{R}_{\geq 0}[x,z]$ is completely log-concave. Prove that for any $\alpha \in [0,1]$ we have
$$b + c \geq \frac{\alpha^\alpha (1-\alpha)^{1-\alpha}}{1 + \alpha(1-\alpha)} \cdot \operatorname{Cap}_{(\alpha, 1-\alpha)}(p).$$
(**Hint:** Follow the proof of the analogous result for real stable p from the lecture notes, altering the proof where needed.)

8. Let $p, q \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ be multiaffine completely log-concave polynomials. Prove that for any $\boldsymbol{\alpha} \in [0,1]^n$ we have
$$\langle p, q \rangle^{\mathbf{1}} \geq \left[\prod_{i=1}^n \frac{\alpha_i^{\alpha_i}(1-\alpha_i)^{1-\alpha_i}}{1 + \alpha_i(1-\alpha_i)}\right] \cdot \operatorname{Cap}_{\boldsymbol{\alpha}}(p) \operatorname{Cap}_{\mathbf{1}-\boldsymbol{\alpha}}(q).$$
Note that $1 + t(1-t) \leq e^t$, so that this bound differs from the real stable bound by at most a simply exponential factor.

9. Prove a linear preservers theorem for completely log-concave polynomials, and using the previous exercise, prove a capacity bounds theorem for linear preservers of completely log-concave polynomials. (**Hint:** The proofs should be very similar to the real stable case.)

10. Generalize the previous exercises to non-multiaffine polynomials, if possible. (**Note:** I have not actually done this myself, and I am not 100% sure it is possible. But I think it should be straightforward.)

11. **Gurvits' conjecture (currently open):** Given d-homogeneous real stable (or even completely log-concave possibly) polynomials $p, q \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ and any $\boldsymbol{\alpha} \in \mathbb{R}^n_{\geq 0}$ such that $\|\boldsymbol{\alpha}\|_1 = d$, show (or find a counterexample) that
$$\sum_{\|\boldsymbol{\kappa}\|_1 = d} \binom{d}{\boldsymbol{\kappa}}^{-1} p_{\boldsymbol{\kappa}} q_{\boldsymbol{\kappa}} \geq \frac{\boldsymbol{\alpha}^{\boldsymbol{\alpha}}}{d^d} \operatorname{Cap}_{\boldsymbol{\alpha}}(p) \operatorname{Cap}_{\boldsymbol{\alpha}}(q).$$
Here, $\binom{d}{\boldsymbol{\kappa}}$ denotes the multinomial coefficient, and the left-hand side of the inequality is the unique (up to scalar) SU_n-invariant inner product on polynomials. (**Note:** In the case that $q(x) = x_1 x_2 \cdots x_n$ with $d = n$, the left-hand side is $\frac{p_{\mathbf{1}}}{d!}$ and for $\boldsymbol{\alpha} = \mathbf{1}$ the right-hand side is $\frac{1}{d^d} \operatorname{Cap}_{\boldsymbol{\alpha}}(p)$, so that this recovers Gurvits' theorem for the permanent.)