HW #2

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1 Lorentzian characterizations

In this section, we consider many possible equivalent characterizations of Lorentzian polynomials of various degrees. In what follows, feel free to use any results we have proven in class.

1.1 Quadratic characterizations

Let A be an $n \times n$ real symmetric matrix with non-negative entries, and let $p(\boldsymbol{x}) := \boldsymbol{x}^\top A \boldsymbol{x}$ be the associated quadratic form. Prove that the following are equivalent (assuming p has non-negative coefficients).

1. p is Lorentzian,
2. p is real stable,
3. p is log-concave (or $\equiv 0$) in the strict positive orthant,
4. the Hessian of $\log p$ is negative semidefinite at some point in the strict positive orthant (or $p \equiv 0$),
5. $p(\boldsymbol{x}t + \boldsymbol{y}) \in \mathbb{R}[t]$ is real-rooted for all $\boldsymbol{x}, \boldsymbol{y}$ in the positive orthant,
6. $p(\boldsymbol{x}t + \boldsymbol{y}) \in \mathbb{R}[t]$ is real-rooted for all $\boldsymbol{x}$ in the positive orthant and all $\boldsymbol{y} \in \mathbb{R}^n$,
7. A has at most one positive eigenvalue,
8. $(\boldsymbol{x}^\top A \boldsymbol{y})^2 \geq (\boldsymbol{x}^\top A \boldsymbol{x})(\boldsymbol{y}^\top A \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y}$ in the positive orthant,
9. $(\boldsymbol{x}^\top A \boldsymbol{y})^2 \geq (\boldsymbol{x}^\top A \boldsymbol{x})(\boldsymbol{y}^\top A \boldsymbol{y})$ for all $\boldsymbol{x}$ in the positive orthant and all $\boldsymbol{y} \in \mathbb{R}^n$,
10. A is negative semidefinite on the orthogonal complement of $A\boldsymbol{x}$ for all $\boldsymbol{x}$ in the positive orthant for which $A\boldsymbol{x} \neq 0$,
11. A is negative semidefinite on the orthogonal complement of $A\boldsymbol{x}$ for some $\boldsymbol{x}$ in the positive orthant,
12. $\det(-A_{S,S}) \leq 0$ for all $S \subseteq [n]$, where $A_{I,J}$ denotes the submatrix of A with rows and columns indexed by I and J respectively.

1.2 General characterizations

Let $p(\boldsymbol{x})$ be a d-homogeneous polynomial in n variables with non-negative coefficients. We let the term *Lorentzian quadratic* refer to a polynomial satisfying any of the equivalent conditions of Section 1.1. We let $D_{\boldsymbol{v}} p$ denote the directional derivative of p in direction $\boldsymbol{v}$, and we let ∂_i denote the partial derivative of p with respect to the variable x_i. Further, we define the *support* of p to be the set of all $\boldsymbol{\kappa} \in \mathbb{Z}^n_{\geq 0}$ such that the $\boldsymbol{x}^{\boldsymbol{\kappa}}$ coefficient of p is non-zero.

We call a polynomial p *indecomposable* if it cannot be written as the sum of two non-zero polynomials which depend on disjoint sets of variables. We call a finite set $M \subset \mathbb{Z}^n_{\geq 0}$ *M-convex* if it satisfies the following exchange axiom: for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in M$ and $i \in [n]$ such that $\alpha_i > \beta_i$, there exists $j \in [n]$ such that $\beta_j > \alpha_j$ and $\boldsymbol{\alpha} - \boldsymbol{e}_i + \boldsymbol{e}_j \in M$. Note that if $M \subseteq \{0,1\}^n$, then M gives the set of bases of a matroid.

When $d \leq 1$, the polynomial p is automatically Lorentzian (and real stable) since p has non-negative coefficients. For $d \geq 2$, prove the following are equivalent (assuming p has non-negative coefficients).

1. p is Lorentzian (or completely log-concave or strongly log-concave, depending on what paper you are reading),
2. $D_{\boldsymbol{v}_1} \cdots D_{\boldsymbol{v}_{d-2}} p$ is a Lorentzian quadratic for all $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d-2}$ in the (strict) positive orthant,
3. (a) the support of p is M-convex, and
   (b) $\partial_{i_1} \cdots \partial_{i_{d-2}} p$ is a Lorentzian quadratic for all $i_1, \ldots, i_{d-2} \in [n]$,
4. (a) $\partial_{i_1} \cdots \partial_{i_k} p$ is indecomposable for all $0 \leq k \leq d-2$ and $i_1, \ldots, i_k \in [n]$, and
   (b) $\partial_{i_1} \cdots \partial_{i_{d-2}} p$ is a Lorentzian quadratic for all $i_1, \ldots, i_{d-2} \in [n]$,
5. $\partial_{i_1} \cdots \partial_{i_k} p$ is log-concave (or $\equiv 0$) in the strict positive orthant for all $0 \leq k$ and $i_1, \ldots, i_k \in [n]$,
6. (a) p is log-concave (or $\equiv 0$) in the strict positive orthant, and
   (b) $\partial_i p$ is Lorentzian for all $i \in [n]$.
7. $(D_{\boldsymbol{v}_1} D_{\boldsymbol{v}_2} D_{\boldsymbol{v}_3} \cdots D_{\boldsymbol{v}_d} p)^2 \geq (D_{\boldsymbol{v}_1} D_{\boldsymbol{v}_1} D_{\boldsymbol{v}_3} \cdots D_{\boldsymbol{v}_d} p) \cdot (D_{\boldsymbol{v}_2} D_{\boldsymbol{v}_2} D_{\boldsymbol{v}_3} \cdots D_{\boldsymbol{v}_d} p)$ for all $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d$ in the (strict) positive orthant,
8. there exists $\boldsymbol{v}$ in the strict positive orthant such that $\partial_{i_1} \cdots \partial_{i_k} D_{\boldsymbol{v}}^{d-2-k} p$ is a Lorentzian quadratic for all $0 \leq k \leq d-2$ and $i_1, \ldots, i_k \in [n]$.

When $d \geq 3$, the following are also equivalent to the above conditions.

9. $D_{\boldsymbol{v}} p$ is Lorentzian for all $\boldsymbol{v}$ in the (strict) positive orthant,
10. (a) p is indecomposable, and
    (b) $\partial_i p$ is Lorentzian for all $i \in [n]$.

2 Other questions on Lorentzian polynomials

You may use the exercises of the previous section in answering these questions.

1. Prove that any homogeneous real stable polynomial (with non-negative coefficients) is Lorentzian.
2. Give an example of a Lorentzian polynomial which is not real stable.
3. Use the mixed volume wikipedia page (https://en.wikipedia.org/wiki/Mixed_volume) to prove that $p(\boldsymbol{x}) = \mathrm{vol}\left(\sum_i x_i K_i\right)$ is Lorentzian for any fixed compact convex sets $K_1, \ldots, K_n \subset \mathbb{R}^d$.
4. Given a matroid M on ground set E and some subset $S \subseteq E$, prove that the number c_k of bases of M containing exactly k elements of S forms an ultra log-concave sequence (with respect to rank).
5. Let $E = \{1, 2, \ldots, n\}$ and let M be any non-empty collection of subsets of E, where each subset has cardinality d. Define $p_M(\boldsymbol{x}) := \sum_{S \in M} \boldsymbol{x}^S$. Prove that p_M is log-concave in the strict positive orthant if and only if M is the set of bases of a matroid. (**Hint:** Is there some way to simulate derivatives via evaluations when a polynomial is multiaffine?)

6. Prove the following Rayleigh-type condition for a Lorentzian polynomial p of degree d:

$$p(\boldsymbol{v}) \cdot \partial_i \partial_j p(\boldsymbol{v}) \leq 2\left(1 - \frac{1}{d}\right) \partial_i p(\boldsymbol{v}) \cdot \partial_j p(\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \mathbb{R}^n_{\geq 0}.$$

   (**Hint:** Consider first the $d = 2$ case, and then: Is there some way to simulate evaluations via derivatives for a general polynomial?)

7. Use the previous exercise to prove a negative dependence property for matroids analogous to spanning trees which we proved in class. That is, compare the probabilities of $i \in E$ being in a given basis of a matroid, conditioned on $j \in E$ being in the basis versus not conditioned on anything. (The constant which appears in this statement is bounded by 2, but it is conjectured for general matroids to be replaceable by $\frac{8}{7}$ [HSW18].)

8. Give examples of Lorentzian polynomials which shows the constant $2\left(1 - \frac{1}{d}\right)$ in the above exercise is tight for all d. (**Hint:** This one is probably hard. The examples can even be chosen to be a family of volume polynomials in 3 variables.)

3 High-dimensional expanders

Given a finite set $E = \{1, 2, \ldots, n\}$, a *simplicial complex* Δ is a non-empty collection of subsets of E for which $T \subseteq S \in \Delta$ implies $T \in \Delta$. We refer to subsets $S \in \Delta$ as the *faces* of Δ. We say that Δ is *pure* if all maximal faces (called *facets*) of Δ, with respect to inclusion, have the same cardinality. The 1-*skeleton* of Δ is the graph with vertices given by the set of one-element faces in Δ and edges given by the set of two-element faces in Δ. Given $S \in \Delta$, the *link of* S is defined as $\Delta^S := \{T \setminus S : S \subseteq T \in \Delta\}$.

Now fix a simplicial complex Δ on E such that $\{i\} \in \Delta$ for all $i \in E$. Let A be the *uniform adjacency matrix* of the 1-skeleton of Δ, defined as the usual adjacency matrix divided by $|E|$. Let D be the *uniform degree matrix* of the 1-skeleton of Δ, defined as the diagonal matrix with diagonal entries given by $A \cdot \mathbf{1}$. Define the *uniform random walk matrix* of Δ via $W := D^{-1}A$. Given any $S \in \Delta$, let A^S, D^S, and W^S denote the respective matrices for the 1-skeleton of the link of S. Note that W^S is always row stochastic matrix and thus gives the transition matrix for a random walk on the 1-skeleton of Δ^S.

Now consider the following definition, which is the specialization of a more general definition seen recently in the literature. See [Lau21] for more details, generalizations, and references (but maybe try the first problem before checking this reference; reading those notes may give some of the answer away).

Definition 3.1 ([Opp18]; see also [Lau21], Definition 19.13 and Exercise 19.16)**.** Let Δ be a pure simpicial complex with facets of size d. We say that Δ is a *uniform* 0*-local spectral expander* if

- the 1-skeleton of Δ^S is connected for all $S \in \Delta$ with $|S| \leq d - 2$, and
- $\lambda_2(W^S) \leq 0$ for all $S \in \Delta$ with $|S| = d - 2$,

where $\lambda_2(W^S)$ is the second largest eigenvalue of the uniform random walk matrix W^S.

As discussed in [Lau21] and the references therein, a corollary of Δ being a local spectral expander is that a certain random walk on the facets of Δ mixes rapidly (with appropriate stationary distribution; here the uniform distribution). See Lap Chi's full course notes at `https://cs.uwaterloo.ca/~lapchi/cs860/notes.html` for more on this topic, but try the following exercises before digging too deeply into the notes.

3.1 Exercises

These were originally proven in [ALGV19].

1. **Warm-up:** Let Δ be the simplicial complex with maximal faces given by the bases of a matroid (i.e., the simplicial complex consisting of the independent sets of the matroid). Prove that Δ is a uniform 0-local spectral expander.

2. **Harder:** Let p be a multiaffine Lorentzian polynomial with $p(\mathbf{1}) = 1$, and let (Δ, w) be the *weighted simplicial complex* with maximal faces given by the support of p and weight function w defined on the maximal faces of Δ via $w(S) := p_S$ (the corresponding coefficient of p). Use the results of [Lau21] to prove that (Δ, w) is a 0-local spectral expander, according to Definition 19.13 of [Lau21].

References

[ALGV19] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant, *Log-concave polynomials ii: High-dimensional walks and an fpras for counting bases of a matroid*, Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, 2019, pp. 1–12.

[HSW18] June Huh, Benjamin Schröter, and Botong Wang, *Correlation bounds for fields and matroids*, arXiv preprint arXiv:1806.02675 (2018).

[Lau21] Lap Chi Lau, *High dimensional expanders*, CS 860: Eigenvalues and Polynomials, U Waterloo, 2021, Available at `https://cs.uwaterloo.ca/~lapchi/cs860/notes/19-HDX.pdf`.

[Opp18] Izhar Oppenheim, *Local spectral expansion approach to high dimensional expanders part i: Descent of spectral gaps*, Discrete & Computational Geometry **59** (2018), no. 2, 293–330.