# Univariate Polynomials

## Polynomial Capacity: Theory, Applications, Generalizations

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# Notation

**Basic polynomial notation:**

- $\mathbb{C}[x] :=$ v.s. of complex polynomials in one variable.
- $\mathbb{C}^d[x] :=$ v.s. of polynomials of degree at most $d$.
- For $p \in \mathbb{C}^d[x]$, we write $p(x) = \sum_{k=0}^{d} p_k x^k$.
- monic $:=$ the leading coeffcient is 1.
- $\deg(p) :=$ the degree of the polynomial.
- $\lambda(p) :=$ the roots/zeros of the polynomial, counting multiplicity.
- $\frac{d}{dx} = \frac{\partial}{\partial x} = \partial_x :=$ derivative with respect to $x$.

**Some other notation for this talk:**

- $\mathbb{C}_h^d[x : y] :=$ v.s. of bivariate homogeneous polynomials of degree $d$.
- For $p \in \mathbb{C}_h^d[x : y]$, we write $p(x : y) = \sum_{k=0}^{d} p_k x^k y^{d-k}$.
- $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} :=$ complex projective line, or Riemann sphere.
- $\mathrm{SL}_2(\mathbb{C}) := 2 \times 2$ invertible complex matrices, $\det = 1$.
- $\mathrm{SU}_2(\mathbb{C}) :=$ subset of unitary matrices in $\mathrm{SL}_2(\mathbb{C})$.

# Outline

1. The big three: roots, coefficients, evaluations
   - Roots and coefficients
   - Real-rooted polynomials
   - Coefficients, evaluations, and log-concavity
2. Interlacing polynomials
   - Interlacing via pictures
   - Classic example: matchings of a graph
3. The Gauss-Lucas theorem and polar derivatives
   - The derivative and complex roots
   - Laguerre's theorem
4. The granddaddy of 'em all: Grace's theorem
   - The apolarity bilinear form
   - Why we care: a preview of next week
   - $SL_2(\mathbb{C})$-invariance
5. Open problems

# Outline

# The big three

The **geometry of polynomials** is generally an investigation of the connections between the various properties of polynomials:

- **Algebraic**, via the roots/zeros of the polynomial.
- **Combinatorial**, via the coefficients of the polynomial.
- **Analytic**, via the evaluations of the polynomial.

**Why do we care?** We use features of the interplay between these three to prove facts about mathematical objects which a priori have nothing to do with polynomials.

**Typical method:**

1. Encode some object as a polynomial which has some nice properties.
2. Apply operations to that polynomial which preserve those properties.
3. Extract information at the end which relates back to the object.

# The fundamental theorem of algebra

**Fundamental theorem of algebra:** For all monic $p \in \mathbb{C}[x]$ with $\deg(p) = d$, there exist $r_1, \ldots, r_d \in \mathbb{C}$ such that

$$p(x) = \sum_{k=0}^{d} p_k x^k = \prod_{i=1}^{d} (x - r_i).$$

**Corollary:** For all $p \in \mathbb{C}_h^d[x:y]$, there exist $(r_1 : s_1), \ldots, (r_d : s_d) \in \mathbb{CP}^1$ such that

$$p(x:y) = \sum_{k=0}^{d} p_k x^k y^{d-k} = \prod_{i=1}^{d} (s_i x - r_i y).$$

**In both cases:** We call these points the *roots of p*, and we will often refer to these two different definitions interchangeably.

**Note:** The $\mathbb{C}_h^d[x:y] \cong \mathbb{C}^d[x]$ case allows roots "at infinity"; consider, e.g.:

$$p(x) = x^k \qquad \text{vs} \qquad p(x:y) = x^k y^{d-k}.$$

# A "converse" to the fundamental theorem

Given $r_1, \dots, r_d \in \mathbb{C}$, there exists a monic polynomial $p \in \mathbb{C}^d[x]$ with roots $r_1, \dots, r_d$ given by

$$p(x) = \prod_{i=1}^{d}(x - r_i) = \sum_{k=0}^{d}(-1)^{d-k} e_{d-k}(r_1, \dots, r_d) x^k$$

where $e_k(r_1, \dots, r_n)$ is the **elementary symmetric polynomial**.

**Important fact:** This is a formula for the coefficients in terms of the roots, but no formula exists in the opposite direction.

**Consolation prize:** The roots are continuous functions of the coefficients.

**Hurwitz's theorem:** A limiting sequence of polynomials with no roots in an open set $U \subset \mathbb{C}$ is either identically zero or has no roots in $U$.

# Real-rooted polynomials

A polynomial $p \in \mathbb{R}[x]$ is **real-rooted** if all of its roots are real. (We also sometimes consider $p \equiv 0$ to be real-rooted.)

**Lemma:** The roots of $p \in \mathbb{R}[x]$ are real or come in conjugate pairs.
**Proof:** $p(x) = \overline{p(\bar{x})}$, where $\bar{x}$ is complex conjugate.

**Special continuity of real roots:** If $p \in \mathbb{R}[x]$ has a simple real root $r_0$, then real perturbations of $p$ will not move $r_0$ off the real line. (Roots can only move off the real line in conjugate pairs.)

**Corollary:** Real perturbations of real-rooted polynomials with simple roots are still real-rooted.

**Corollary:** The set of real-rooted polynomials in $\mathbb{R}^d[x]$ is equal to the closure of its interior (which is non-empty).

# Linear preservers of real-rootedness

**Rolle's theorem for polynomials:** Between any two zeros of a polynomial $p \in \mathbb{R}[x]$, there is at least one zero of $\partial_x p$.

**Corollary:** If $p \in \mathbb{R}[x]$ has only real roots, then $\partial_x p$ has only real roots.
**Proof:** Apply Rolle's theorem to each consecutive pair of roots.

**In modern language:** The linear operator $\partial_x$ preserves real-rootedness.

**What about other linear preservers?**

1. Shifted derivative: $p \mapsto p + \alpha \partial_x p$ for $\alpha \in \mathbb{R}$.
2. Scaling: $p \mapsto p(rx)$ for $r \in \mathbb{R}$.
3. Inversion: $p \mapsto x^d \cdot p(x^{-1})$, when $p \in \mathbb{R}^d[x]$.
4. $SL_2(\mathbb{R})$-action: act on the roots by real Möbius transformation. (This is a linear action! We will discuss this in more detail.)

**Note:** Last two preservers rely on a choice of degree, which is equivalent to specifying multiplicity of the root at infinity.

# Newton's inequalities

**Newton's inequalities:** If $p \in \mathbb{R}[x]$ is real-rooted and $\deg(p) = d$, then

$$\frac{p_0}{\binom{d}{0}}, \frac{p_1}{\binom{d}{1}}, \frac{p_2}{\binom{d}{2}}, \ldots, \frac{p_d}{\binom{d}{d}}$$

is a log-concave sequence ($c_k^2 \geq c_{k-1}c_{k+1}$). This condition is called **ultra log-concave (ULC)**, and if $p_k > 0$, this implies log-concave and unimodal.

**Proof:** First note that writing $p(x) = \sum_{k=0}^{d} \binom{d}{k} \tilde{p}_k x^k$ gives

$$\frac{\partial_x p}{d} = \sum_{k=0}^{d-1} \binom{d-1}{k} \tilde{p}_{k+1} x^k \quad \text{and} \quad x^d \cdot p(x^{-1}) = \sum_{k=0}^{d} \binom{d}{k} \tilde{p}_{d-k} x^k,$$

and both preserve real-rootedness. Apply these operators to reduce to

$$\sum_{k=0}^{2} \binom{2}{k} \tilde{p}_{k+j} x^k$$

for any $j$. Real-rootedness implies $0 \leq b^2 - 4ac = 4(\tilde{p}_{j+1}^2 - \tilde{p}_j \tilde{p}_{j+2})$.

# A little more on the proof of Newton's inequalities

**How exactly do we reduce to quadratics?**
Let's consider everything in terms of $p \in \mathbb{R}_h^d[x : y]$.

**First:** $\partial_x$ is the "same" in all of $\mathbb{R}[x]$, $\mathbb{R}^d[x]$, and $\mathbb{R}_h^d[x : y]$.

**How does $\partial_y$ for $p = \sum_k p_k x^k y^{d-k}$ translate to $p \in \mathbb{R}^d[x]$?**

1. First flip coefficients: $p \mapsto x^d \cdot p(x^{-1})$.
2. Now apply usual derivative: $p \mapsto \partial_x p$.
3. Finally, flip coefficients back with new degree: $p \mapsto x^{d-1} \cdot p(x^{-1})$.

**So $\partial_y$ preserves real-rootedness.**
Further, we can now easily reduce to quadratics in $\mathbb{R}_h^d[x : y]$ via:

$$\frac{2}{d!}\partial_x^j \partial_y^{d-2-j} \left[\sum_{k=0}^{d} \binom{d}{k} \tilde{p}_k x^k y^{d-k}\right] = \sum_{k=0}^{2} \binom{2}{k} \tilde{p}_{k+j} x^k y^{2-k}.$$

# Some "converses" to Newton's inequalities

**Kurtz '92:** For $p(x) = \sum_{k=0}^d p_k x^k$, if we have

$$p_k^2 \geq 4 p_{k-1} p_{k+1} \qquad \text{for all valid } k,$$

then $p$ is real-rooted. (Discriminant condition when $d = 2$.)

**An actual converse:** For $p(x:y) \in \mathbb{R}_h^d[x:y]$ with $\geq 0$ coefficients, the following are equivalent.

1. The coefficients of $p$ form an ultra log-concave sequence.
2. $\partial_x^i \partial_y^j p(x:y)$ is log-concave as a bivariate function on $\mathbb{R}_+^2$ (the positive orthant) for all valid $i, j$.
3. $\partial_x^i \partial_y^{d-i-2} p(x:y)$ is a real-rooted quadratic for all valid $i$, and the coefficient sequence of $p$ has no "internal zeros".

**Corollary:** Real-rooted polynomials are log-concave in the positive orthant.

**Foreshadowing:** Proof will go via Lorentzian polynomials.

# The method, revisited

**The method:**

1. Encode some object as a polynomial which has some nice properties.
   - Real-rootedness, ULC coefficients, log-concavity, etc.
2. Apply operations to that polynomial which preserve those properties.
   - Derivatives, $SL_2(\mathbb{R})$, **others?**
3. Extract information at the end which relates back to the object.
   - Coefficients, evaluations, log-concavity, **capacity**, etc.

**Classic examples:**

- Graph polynomials where coefficients count things (matching polynomial, spanning tree polynomial).
- Polynomials where evaluations count things (chromatic polynomial, Ehrhart polynomial).
- Other generating functions (Schur polynomials, contingency tables).

# Outline

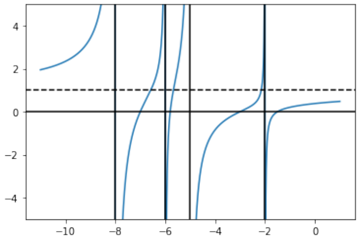
# Interlacing roots

Given real-rooted polynomials $p, q \in \mathbb{R}[x]$ with positive leading coefficients, we say that $p$ **interlaces** $q$ and write $p \ll q$ if

$$\cdots \leq \lambda_2(p) \leq \lambda_2(q) \leq \lambda_1(p) \leq \lambda_1(q),$$

where $\lambda_n(p) \leq \cdots \leq \lambda_1(p)$ are the ordered roots of $p$. **This is a closed property**, and implies $\deg(q) - \deg(p) \in \{0, 1\}$. If all inequalities are strict, we say that $p$ **strictly interlaces** $q$. **This is the interior.**

**Key picture** is the graph of $\frac{q(x)}{p(x)}$. E.g. $\frac{(x+7)(x+5.8)(x+3)(x+1.5)}{(x+8)(x+6)(x+5)(x+2)}$:



# Characterization of interlacing polynomials

Let $p, q \in \mathbb{R}^d[x]$ be monic with $d$ simple roots such that $p, q$ don't share any roots. (True more generally, but this is simpler.)

**Theorem (Hermite-Kakeya-Obreschkoff):** The following are equivalent.

- $p \ll q$ (that is, $\cdots < \lambda_2(p) < \lambda_2(q) < \lambda_1(p) < \lambda_1(q)$).
- $\text{sgn}(p(\lambda_i(q))) = (-1)^{i-1}$ for all $i$.
- $ap + bq$ is real-rooted for all $a, b \in \mathbb{R}$.
- $W(p, q) = p \cdot \partial_x q - q \cdot \partial_x p \geq 0$ on $\mathbb{R}$.

**Corollary:** $p \mapsto p + \alpha \partial_x p$ preserves real-rootedness for $\alpha \in \mathbb{R}$.
**Proof:** $\partial_x p \ll p$ by Rolle's theorem.

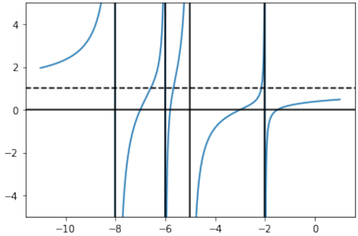
**Corollary:** If $p \ll q$ and $p \ll r$, then $p \ll aq + br$ for all $a, b \geq 0$.
(The polynomials $q$ and $r$ have a **common interlacer**.)
**Proof:** E.g., bilinearity of Wronskian $W(p, q)$.

# Proof of characterization

**Theorem:** The following are equivalent.

- $p \ll q$.
- $ap + bq$ is real-rooted for all $a, b \in \mathbb{R}$.
- $W(p, q) = p \cdot \partial_x q - q \cdot \partial_x p \geq 0$ on $\mathbb{R}$.

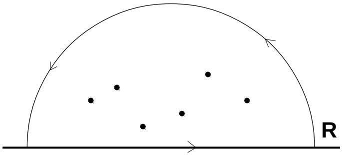
**Proof by picture:** $ap + bq = 0 \iff \frac{q}{p} = -\frac{a}{b}$.




$$\partial_x \left[\frac{q}{p}\right] = \frac{p \cdot \partial_x q - q \cdot \partial_x p}{p^2} = \frac{W(p, q)}{p^2}$$

# The Hermite-Biehler theorem

**Theorem (Hermite-Biehler):** Given monic $p, q \in \mathbb{R}^d[x]$ with $d$ roots, we have $p \ll q$ strictly iff $p + iq$ has all its roots in the upper half-plane.
**Proof by picture:** Consider winding number of

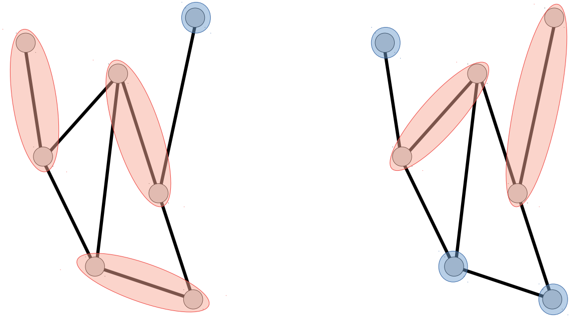



**Along real line:** $(\mathrm{Re}, \mathrm{Im}) = \ldots, (-,+), (-,-), (+,-), (+,+) = \frac{d}{2}$ loops.
**Along arc:** $p + iq \approx (1+i)x^d \implies$ half of circle gives half of $d$ loops.

**Corollary:** If $p$ has simple real roots, then $\partial_x p + ip$ has all its roots in the upper half-plane.

# Classic example: matchings of a graph

Given a graph $G = (V, E)$, a $k$-**matching** $M$ of $G$ is a selection of $k$ edges for which no two edges touch the same vertex.



**Matching polynomial:** $M_G(x) := \sum_{k=0}^{\lfloor |V|/2 \rfloor} (-1)^k m_k x^{|V|-2k}$.

# Classic example: matchings of a graph

**Matching polynomial:** $M_G(x) := \sum_{k=0}^{\lfloor |V|/2 \rfloor} (-1)^k m_k x^{|V|-2k}$.

**Classic theorem (Heilmann-Lieb '72):** For any graph $G$, the matching polynomial has only real roots.

**Corollary:** The sequence $m_k$ is ultra log-concave (log-concave, unimodal).
**Proof:** $M_G(x)$ real-rooted iff $\mu_G(x) := \sum_k m_k x^k$ is real-rooted. **Why?**

1. $M_G(x) = x^{|V|} \cdot \sum_{k=0}^{\lfloor |V|/2 \rfloor} m_k \cdot (-x^{-2})^k = x^{|V|} \cdot \mu_G(-x^{-2})$.
2. $\mu_G(x) = \prod_{i=1}^{\lfloor |V|/2 \rfloor} (x + r_i) \iff M_G(x) = x^{|V| \mod 2} \prod_{i=1}^{\lfloor |V|/2 \rfloor} (r_i x^2 - 1)$.
3. Roots of $M_G(x)$ come in $\pm$ pairs (except at $x = 0$).

# Proof of the Heilmann-Lieb theorem

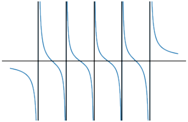
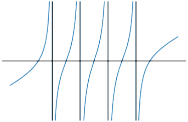
**Matching polynomial:** $M_G(x) := \sum_{M \in G} x^{|V|-2|M|} = \sum_{k=0}^{\lfloor |V|/2 \rfloor} m_k x^{|V|-2k}$.

**Proof:** Induction on interlacing relation $M_{G \setminus v} \ll M_G$.

1. Recurrence relation for $M_G(x)$ based on subgraphs, for any $v \in V$:

$$M_G(x) = x \cdot M_{G \setminus v}(x) - \sum_{u \sim v} M_{G \setminus uv}(x).$$

2. Divide through by $M_{G \setminus v}$: $\dfrac{M_G(x)}{M_{G \setminus v}(x)} = x - \sum_{u \sim v} \dfrac{M_{G \setminus uv}(x)}{M_{G \setminus v}(x)}$.

3. $\partial_x \left[ x - \sum_{u \sim v} \right.$  $\left. \right] \geq 0 \implies \dfrac{M_G(x)}{M_{G \setminus v}(x)} =$  .

# Outline

# The Gauss-Lucas theorem

**Theorem (Gauss-Lucas):** If the roots of $p \in \mathbb{C}[x]$ are contained in a convex set $K \subset \mathbb{C}$, then the roots of $\partial_x p$ are also contained in $K$.
**Proof:** By the product rule for $\partial_x$, if $\partial_x p(r) = 0$ then

$$0 = \overline{\frac{\partial_x p}{p}(r)} = \overline{\frac{\sum_{i=1}^d \prod_{j \neq i}(r - \lambda_j(p))}{\prod_{i=1}^d (r - \lambda_i(p))}} = \overline{\sum_{i=1}^d \frac{1}{r - \lambda_i(p)}} = \sum_{i=1}^d \frac{r - \lambda_i(p)}{|r - \lambda_i(p)|^2}.$$

Rearranging then gives

$$\sum_{i=1}^d \frac{r}{|r - \lambda_i(p)|^2} = \sum_{i=1}^d \frac{\lambda_i(p)}{|r - \lambda_i(p)|^2} \implies r = \sum_{i=1}^d \frac{\frac{1}{|r-\lambda_i(p)|^2}}{\sum_{j=1}^d \frac{1}{|r-\lambda_j(p)|^2}} \cdot \lambda_i(p).$$

This is a convex combination of the $\lambda_i(p)$.

**Conceptual proof:** $0 = \sum_i \frac{r-\lambda_i(p)}{|r-\lambda_i(p)|^2} \iff$ equillibria of electric potential.

**Corollary:** $\partial_x$ preserves real-, half-plane-, or disc-rooted polynomials.

# The polar derivative

**Recall:** Both $\partial_x$ and $\partial_y$ for $p \in \mathbb{R}_h^d[x:y]$ preserve real-rootedness.
**Gauss-Lucas:** $\partial_x$ preserves other sets.

**What about $\partial_y$?**

$$\partial_y = (\text{flip coeff.})^{-1} \circ \partial_x \circ (\text{flip coeff.})$$

Since flipping coeff. $\iff$ inverting roots, $\partial_y$ preserves roots in sets which are inverses of convex sets. **E.g.:** half-planes, exterior of discs.

**More generally:** $\partial_y$ preserves roots in any set "convex with respect to 0".

**Why?** Flipping coefficients/inverting roots maps $0 \longleftrightarrow \infty$.

Flipping coefficients is a Möbius transformation. **What about others?**

- If $\phi$ is a Möbius transform, then $\phi^{-1} \circ \partial_x \circ \phi$ preserves roots in $\phi^{-1}(K)$ for any convex $K$. **So what?**

# More general polar derivatives via Möbius transformations

**Möbius transformation:** $\phi \in \mathrm{SL}_2(\mathbb{C})$ and $\phi : \mathbb{CP}^1 \to \mathbb{CP}^1$.
**On polynomials, $\phi$ acts on roots:** For $p \in \mathbb{C}_h^d[x : y]$,

$$\phi^{-1} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \implies \phi \cdot p = p\left(\phi^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right) = p(\alpha x + \beta y : \gamma x + \delta y).$$

**Lemma:** Given $\phi^{-1}$ as above and $p \in \mathbb{C}_h^d[x : y]$, we have

$$(\phi^{-1} \circ \partial_x \circ \phi)p = (\alpha \partial_x + \gamma \partial_y)p.$$

**Proof:**

$$\begin{aligned}(\phi^{-1} \circ \partial_x \circ \phi)p &= \phi^{-1} \cdot \partial_x[p(\alpha x + \beta y : \gamma x + \delta y)] \\ &= \phi^{-1} \cdot [(\alpha \partial_x + \gamma \partial_y)p](\alpha x + \beta y : \gamma x + \delta y) \\ &= (\alpha \partial_x + \gamma \partial_y)p.\end{aligned}$$

# Laguerre's theorem

**Laguerre's theorem:** $(\alpha\partial_x + \gamma\partial_y)$ preserves roots in circular regions $C \subset \mathbb{CP}^1$ which do not contain $(\alpha : \gamma)$.

**Circular regions:** Equivalent definitions.

- Möbius transformations of the unit disc.
- Half-spaces in $\mathbb{R}^3$ intersected with the Riemann sphere ($\cong \mathbb{CP}^1$).

**E.g.:** Discs, exteriors of discs (with $\infty$), half-planes.

**Proof:** By the lemma, $\alpha\partial_x + \gamma\partial_y = \phi^{-1}\partial_x\phi$ with $\phi^{-1}(\infty) = (\alpha : \gamma)$.

1. All roots of $p$ are contained in $C \subset \mathbb{CP}^1$ with $(\alpha : \gamma) \notin C$.
2. Since $\phi$ acts on roots, roots of $\phi \cdot p$ are contained in $\phi \cdot C \in \mathbb{CP}^1$ with $\infty = \phi(\alpha : \gamma) \notin \phi \cdot C$.
3. Therefore $\phi \cdot C \subset \mathbb{C}$ is convex and Gauss-Lucas applies.
4. Finally, the inverse action $\phi^{-1}$ moves roots back to $C \subset \mathbb{CP}^1$.

# Polar derivatives and real-rooted polynomials

**Fact:** If $p$ is real-rooted, then $(\alpha\partial_x + \gamma\partial_y)p$ is real-rooted for all $\alpha, \gamma \in \mathbb{R}$.
**Proof:** $\alpha\partial_x + \gamma\partial_y = \phi^{-1}\partial_x\phi$, and we can choose $\phi \in \mathrm{SL}_2(\mathbb{R})$.

**Corollary:** The roots of $\partial_x p$ and $\partial_y p$ are interlaced (by Hermite-Kakeya-Obreschkoff theorem).

**Fact:** If $p \ll q$, then $L := \alpha\partial_x + \gamma\partial_y$ implies $Lp \ll Lq$ for all $\alpha, \gamma \in \mathbb{R}$.
**Proof:** $p \ll q$

1. $\implies$ $p + iq$ has all roots in the closure of upper half-plane $= \overline{\mathcal{H}_+}$ (by Hermite-Biehler theorem),
2. $\implies$ $L(p + iq) = Lp + iLq$ has all roots in $\overline{\mathcal{H}_+}$,
3. $\implies$ $Lp \ll Lq$ (by Hermite-Biehler again).

**Further:** Real linear preservers of roots in $\overline{\mathcal{H}_+}$ preserve interlacing.

# Outline

# A special bilinear form on polynomials

**Laguerre:** If $(r_1 : s_1), \ldots, (r_d : s_d) \in C$ and $(\alpha_1 : \gamma_1) \notin C$, then

$$r(x:y) = (\alpha_1 \partial_x + \gamma_1 \partial_y) \prod_{i=1}^{d} (s_i x - r_i y) \qquad \text{has all roots in } C.$$

**Induct:** If $(r_i : s_i) \in C$ and $(\alpha_i : \gamma_i) \notin C$, then

$$r(x:y) = \prod_{i=1}^{d'} (\alpha_1 \partial_x + \gamma_1 \partial_y) \prod_{i=1}^{d} (s_i x - r_i y) \qquad \text{has all roots in } C.$$

If $d' \leq d$, the root conditions guarantee that $r \not\equiv 0$.

**Candidate for "interesting" bilinear form** for $p, q \in \mathbb{C}_h^d[x:y]$:

$$\langle p, q \rangle^d := p(\partial_y : -\partial_x) q(x:y) \in \mathbb{C}.$$

Also can be defined for $\mathbb{C}^d[x]$. **In terms of coefficients?**

# Grace's apolarity theorem

**Theorem (Grace):** If $p, q \in \mathbb{C}^d[x]$ and a circular region $C$ are such that the roots of $q$ are all in $C$ and the roots of $p$ are all not in $C$, then

$$\sum_{k=0}^{d} \binom{d}{k}^{-1} (-1)^k p_k q_{d-k} \neq 0.$$

**Proof:** If we can show that this bilinear form is equal up to scalar the one from the previous slide, then the previous slide proves the theorem. On the monomial basis, we compute:

$$\begin{aligned}
\langle x^k y^{d-k}, x^j y^{d-j} \rangle^d &= (-1)^{d-k} \partial_y^k \partial_x^{d-k} x^j y^{d-j} \\
&= (-1)^{d-k} k!(d-k)! \cdot \delta_{j=d-k} \\
&= (-1)^d d! \left[ \binom{d}{k}^{-1} (-1)^k \cdot \delta_{j=d-k} \right].
\end{aligned}$$

# Grace's theorem: Why do we care? (A preview)

Some bilinear form is non-zero. **So what?**

Many classical theorems are proven using Grace's theorem. **How?**

**First:** We can interpret the bilinear form as a choice of isomorphism between $\mathbb{C}_h^d[x:y]$ and its dual space $\mathbb{C}_h^d[x:y]^*$ via $p \longleftrightarrow \langle p, \cdot \rangle^d$.

**Next:** Induce a map from linear operators on polynomials to polynomials in more variables. Letting $\mathcal{L}_d$ denote the space of operators,

$$\mathcal{L}_d \cong \mathbb{C}_h^d[x:y] \otimes \mathbb{C}_h^d[x:y]^* \overset{\downarrow}{\cong} \mathbb{C}_h^d[x:y] \otimes \mathbb{C}_h^d[x:y] \cong \mathbb{C}_h^{(d,d)}[x:y, z:w].$$

**Denoting this by** $\text{Symb}^d : \mathcal{L}_d \xrightarrow{\sim} \mathbb{C}_h^{(d,d)}[x:y, z:w]$ gives:

$$T[p](x:y) = \left\langle \text{Symb}^d[T](x:y, z:w), p(z:w) \right\rangle^d.$$

**Finally:** Zero location of $p$ and $\text{Symb}^d[T]$ implies non-vanishing of $T[p]$.

# $\mathrm{SL}_2(\mathbb{C})$ and the apolarity form

**Fact:** The apolarity form $\langle p, q \rangle^d$ is **uniquely** $\mathrm{SL}_2(\mathbb{C})$-invariant:

$$\langle p, q \rangle^d = \langle \phi \cdot p, \phi \cdot q \rangle^d \qquad \text{for all } p, q, \phi.$$

Makes sense, as the apolarity theorem is $\mathrm{SL}_2(\mathbb{C})$ invariant.

**A special $\mathrm{SL}_2(\mathbb{C})$-invariant operator $D$:**

$$D\left[p(x:y)q(z:w)\right] := (\partial_x \partial_w - \partial_y \partial_z)\left[p(x:y)q(z:w)\right].$$

**Corollary(?):** $D^d[pq] = \langle p, q \rangle^d$ up to scalar.
**Proof:** Uniqueness of $SL_2(\mathbb{C})$-invariant bilinear form, or easy computation.

**Stronger:** The $D$ map preserves **multivariate** root location properties for polynomials $p(x:y, z:w)$. (The $D$ map acts on the space $\mathbb{C}_h^{(d,d)}[x:y, z:w]$ of polynomials taking input in $\mathbb{CP}^1 \times \mathbb{CP}^1$.)

# An aside: Representation theory of $\mathrm{SL}_2(\mathbb{C})$

**Theorem:** The finite dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{C})$ are precisely given by $V_d := \mathbb{C}_h^d[x:y]$ for all $d \geq 0$.

**Theorem (Clebsch-Gordon):** The tensor square decomposes as

$$\mathbb{C}_h^{(d,d)}[x:y,z:w] \cong V_d \otimes V_d \cong V_{2d} \oplus V_{2d-2} \oplus V_{2d-4} \oplus \cdots \oplus V_2 \oplus V_0.$$

**Fact:** The $D = \partial_x \partial_w - \partial_y \partial_z$ map acts as $D : V_d \otimes V_d \to V_{d-1} \otimes V_{d-1}$. $\mathrm{SL}_2(\mathbb{C})$-invariance implies $D$ simply projects away from the top component in the above decomposition (the $V_{2d}$ component).

**Other names:** Cayley's $\Omega$ process, transvectants, Reynolds operator, etc.

**Corollary(?):** The $V_0$ component picks out the apolarity form.
**Proof:** Uniqueness of $\mathrm{SL}_2(\mathbb{C})$-invariant bilinear form. (The decomposition is also itself a proof of uniqueness.)

# Outline

# Sendov's conjecture

Suppose $p \in \mathbb{C}[x]$ has all its roots $\lambda_1(p), \ldots, \lambda_d(p)$ in the closed unit disc. **How far away can the critical points be?**

**Conjectural worst case:** $p(x) = x^d - 1 \implies |\lambda_i(p) - \lambda_j(\partial_x p)| = 1.$

**Sendov's conjecture:** $\forall i \; \exists j$ such that $|\lambda_i(p) - \lambda_j(\partial_x p)| \leq 1.$ That is, every zero of $p$ is within distance 1 of a critical point of $p$.

**Known facts:**

- Known to be true for $d \leq 8$.
- For $d = 3$, critical points are the foci of the inscribed ellipse of the convex hull of the zeros.
- If $|\lambda_i(p)| = 1$, then the bound is known for that choice of $i$.
- For any particular fixed root $r_0$, there is a number $d_0$ for which $d \geq d_0$ implies the bound for $\lambda_i(p) = r_0$.
- The conjectural worst case is **not** the only local maximum.

# Apolarity theorem for $\mathrm{SU}_n(\mathbb{C})$ form

**Grace's theorem:** Non-vanishing for $\mathrm{SL}_2(\mathbb{C})$-invariant bilinear form.

**Can we extend this beyond $\mathrm{SL}_2(\mathbb{C})$?** There isn't quite an $\mathrm{SL}_n(\mathbb{C})$-invariant form, but there is an $\mathrm{SU}_n(\mathbb{C})$-invariant form for polynomials $p \in \mathbb{C}_h^d[x_1 : \cdots : x_n]$ with $p(x) = \sum_{\boldsymbol{\mu}} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$:

$$\langle p, q \rangle^d := \sum_{|\boldsymbol{\mu}|=d} \binom{d}{\boldsymbol{\mu}}^{-1} p_{\boldsymbol{\mu}} q_{\boldsymbol{\mu}}.$$

**Open question:** For what classes of polynomials do we get a Grace-type theorem for this bilinear form?

**Alternative form:** $\langle p, q \rangle^d = D^d(p(\boldsymbol{x})q(\boldsymbol{z}))$ for $D := \sum_{i=1}^{n} \partial_{x_i} \partial_{z_i}$.

**Another idea:** Extend to multivariate setting via $\mathrm{SL}_2(\mathbb{C})^n$ (next week).