Univariate Polynomials Exercises

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In the lecture period, we only were able to get through the slides up until Laguerre's theorem was proven and discussed. Some of the following questions follow somewhat straightforwardly from the material of the later slides that we did not get to in the lecture. These are marked by the symbol *.

Exercises

1. A multivariate polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ is said to be *stable* if $p(\boldsymbol{x}) \neq 0$ whenever $x_1, \ldots, x_n$ are all in the complex upper half-plane. Prove that the elementary symmetric polynomials are stable only using the univariate theory.

2. Let $p(x) = 1 + x^d + \sum_{k \in S} c_k x^k \in \mathbb{C}^d[x]$ be a polynomial where S is some subset of $\{1, 2, ..., d-1\}$. Give a tight bound on the size of S which guarantees p is not real-rooted, with no constraints on the coefficients.

3. Let $p_1, \ldots, p_m, q \in \mathbb{R}^d[x]$ be real-rooted monic polynomials such that $q \ll p_i$ for all i. First show that every non-negative linear combination of p_i is real-rooted. Next, define maps $\phi_k : \mathbb{R}_+^m \to \mathbb{R}$ (where $\mathbb{R}_+^m$ denotes the closed positive orthant, excluding the $\mathbf{0}$ vector) via
$$\phi_k : (a_1, \ldots, a_m) \mapsto \lambda_k \left(\sum_{i=1}^{m} a_i p_i \right),$$
where $\lambda_k(p)$ denotes the k^{th} largest root of p. Describe the image of ϕ_k in terms of the p_i for each k. (Hint: Try to do it first for $k = 1$.)

4. Given a complex polynomial p with all roots on the unit circle, prove that up to scalar the coefficients satisfy $p_k = \bar{p}_{d-k}$ where d is the degree of p. What is the analogous result if "unit circle" is replaced by "real line", and how are these two results related?

5. For any real-rooted polynomial p, prove that either $\partial_x p \ll \partial_y p$ or $\partial_y p \ll \partial_x p$. Give a condition which determines which direction the interlacing relation goes.

6. *Construct a bilinear form B on $\mathbb{C}_h^d[x : y]$ which has the following property: If $p, q \in \mathbb{C}_h^d[x : y]$ have no roots in the upper half-plane, then $B(p, q) \neq 0$. (Hint: Is there a natural way to turn a polynomial into a linear operator on polynomials?)

7. *Let p, q be two monic real-rooted polynomials of degree d which have simple roots and don't share any roots. If $p \ll q$, then $p + iq$ has all its roots in the upper half-plane. This is one direction of the Hermite-Biehler theorem. (Hint: Use the argument principle.)

8. *(This problem requires some basic invariant theory.) Consider the standard action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}_h^d[x : y]$, and further let $\mathrm{SL}_2(\mathbb{C})$ act on $\mathbb{C}_h^d[x : y] \otimes \mathbb{C}_h^d[x : y]$ via the diagonal action. Show that $\partial_x \otimes \partial_y - \partial_y \otimes \partial_x$ is $\mathrm{SL}_2(\mathbb{C})$-invariant, and describe how this operator acts in terms of the irreducible components of $\mathbb{C}_h^d[x : y] \otimes \mathbb{C}_h^d[x : y]$.