Applications of Stable Polynomials

Polynomial Capacity: Theory, Applications, Generalizations

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Notation

Polynomial notation:

- $\mathbb{R}, \mathbb{R}_+, \mathbb{C} :=$ real, non-negative real, complex numbers.
- $\mathbb{C}[x] :=$ v.s. of complex polynomials in one variable.
- $\mathbb{C}^d[x] :=$ v.s. of polynomials of degree at most d.
- For $p \in \mathbb{C}^d[x]$, we write $p(x) = \sum_{k=0}^d p_k x^k$.
- $\boldsymbol{x}^{\boldsymbol{\mu}} := \prod_i x_i^{\mu_i}$, $\binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}} := \prod_i \binom{\lambda_i}{\mu_i}$, and $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ is entrywise.
- $\mathbb{C}[\boldsymbol{x}] :=$ v.s. of complex polynomials in n variables.
- $\mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}] :=$ v.s. of polynomials of degree at most λ_i in x_i.
- For $p \in \mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$, we write $p(\boldsymbol{x}) = \sum_{\boldsymbol{0} \leq \boldsymbol{\mu} \leq \boldsymbol{\lambda}} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$.
- $\mathbb{R}[\boldsymbol{x}] :=$ v.s. of real polynomials in n variables.
- $\mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}] :=$ v.s. of real polynomials of degree at most λ_i in x_i.
- $\frac{d}{dx} = \frac{\partial}{\partial x} = \partial_x :=$ derivative with respect to x.
- $p(\boldsymbol{a} \cdot t + \boldsymbol{b}) = p(a_1 t + b_1, \ldots, a_n t + b_n) \in \mathbb{C}^{\lambda_1 + \cdots + \lambda_n}[t]$ is a **linear restriction** of the polynomial $p \in \mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$, where $\boldsymbol{a} \in \mathbb{R}_+^n$ and $\boldsymbol{b} \in \mathbb{R}^n$.

Recall: The big three

The **geometry of polynomials** is generally an investigation of the connections between the various properties of polynomials:

- **Algebraic**, via the roots/zeros of the polynomial.
- **Combinatorial**, via the coefficients of the polynomial.
- **Analytic**, via the evaluations of the polynomial.

Why do we care? We use features of the interplay between these three to prove facts about mathematical objects which a priori have nothing to do with polynomials.

Typical method:

1. Encode some object as a polynomial which has some nice properties.
2. Apply operations to that polynomial which preserve those properties.
3. Extract information at the end which relates back to the object.

Outline

1. The Borcea-Brändén characterization
   - Real stable polynomials
   - The BB characterization
   - Link between complex and real cases
2. Application: Multivariate matching polynomial
   - Univariate to multivariate
   - Real stability via the multiaffine part operator
   - Another proof of real stability
3. Application: Spanning tree polynomial
   - Real stability of the spanning tree polynomial
   - The strong Rayleigh conditions
   - Connection to matroids
4. Open problems

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Real stable polynomials

Recall: A polynomial $p \in \mathbb{C}[\boldsymbol{x}]$ is said to be **stable** if

$$p(x_1, \ldots, x_n) \neq 0 \quad \text{whenever} \quad x_i \in \mathcal{H}_+ := \text{ complex upper half-plane.}$$

If p has real coefficients, then it is said to be **real stable**.
Equivalent: p real stable if $p(\boldsymbol{x}) \neq 0$ whenever $\boldsymbol{x} \in \mathcal{H}_-^n$ (lower half-plane).

Important properties for real stable $p \in \mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$:

1. **Generalizes real-rootedness:** If p is univariate, then p is real-rooted.
2. **Linear restrictions:** $p(\boldsymbol{a} \cdot t + \boldsymbol{b}) \in \mathbb{R}^{\lambda_1 + \cdots + \lambda_n}[t]$ is real-rooted for all $\boldsymbol{a} \in \mathbb{R}_+^n$ (positive orthant) and $\boldsymbol{b} \in \mathbb{R}^n$.
3. **Strong Rayleigh [Brändén '07]:** For p multiaffine ($\boldsymbol{\lambda} = \boldsymbol{1}$),

$$R_{ij}(p) := \partial_{x_i} p \cdot \partial_{x_j} p - p \cdot \partial_{x_i} \partial_{x_j} p \geq 0 \quad \text{for } x \in \mathbb{R}^n, \text{ all } i, j.$$

4. **Multiaffine equivalence:** Walsh coincidence theorem says real stability of p is equivalent to that of its multiaffine **polarization**.

The Borcea-Brändén characterization (stable)

Definition: The **symbol** of a linear operator $T : \mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}] \to \mathbb{C}[\boldsymbol{x}]$:

$$\mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x}, \boldsymbol{z}) := T\left[\prod_{i=1}^{n}(x_i + z_i)^{\lambda_i}\right] = \sum_{\boldsymbol{0} \leq \boldsymbol{\mu} \leq \boldsymbol{\lambda}} \binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}} \boldsymbol{z}^{\boldsymbol{\lambda}-\boldsymbol{\mu}} T[\boldsymbol{x}^{\boldsymbol{\mu}}].$$

Here T acts only on $\boldsymbol{x}$, $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ is entrywise, and $\binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}} := \prod_i \binom{\lambda_i}{\mu_i}$.

Theorem (Borcea-Brändén '09)

For a given linear operator $T : \mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}] \to \mathbb{C}[\boldsymbol{x}]$, we have that T preserves stability (allowing $\equiv 0$) if and only if one of the following holds:

1. $\mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x}, \boldsymbol{z})$ *is stable.*
2. *The image of T is a one-dimensional space of stable polynomials.*

Conceptual takeaway: T preserves stability "iff" its symbol is stable.

The Borcea-Brändén characterization (real stable)

Definition: The **symbol** of a linear operator $T : \mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}] \to \mathbb{R}[\boldsymbol{x}]$:

$$\mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x}, \boldsymbol{z}) := T\left[\prod_{i=1}^{n} (x_i + z_i)^{\lambda_i}\right] = \sum_{\boldsymbol{0} \le \boldsymbol{\mu} \le \boldsymbol{\lambda}} \binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}} \boldsymbol{z}^{\boldsymbol{\lambda}-\boldsymbol{\mu}} T[\boldsymbol{x}^{\boldsymbol{\mu}}].$$

Theorem (Borcea-Brändén '09)

For a given linear operator $T : \mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}] \to \mathbb{R}[\boldsymbol{x}]$, we have that T preserves real stability (allowing $\equiv 0$) if and only if one of the following holds:

1. $\mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x}, \boldsymbol{z})$ *is real stable.*
2. $\mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x} \cdot \boldsymbol{z}, \boldsymbol{1})$ *is real stable (where $\boldsymbol{x} \cdot \boldsymbol{z}$ is entrywise).*
3. *The image of T is a two-dimensional space of real stable polynomials.*

Two conditions now? Real stable iff $\mathcal{H}_+^n$-stable iff $\mathcal{H}_-^n$-stable.

1. $\mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x}, \boldsymbol{z})$: preserves $\mathcal{H}_+^n$-stability by previous theorem.
2. $\mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x} \cdot \boldsymbol{z}, \boldsymbol{1})$: maps $\mathcal{H}_-^n$-stable to $\mathcal{H}_+^n$-stable by prev. theorem.

Link between complex and real cases

For the real case: Need a link between real stability and complex stability.

Recall: In the univariate case, the following are equivalent.

1. (Interlacing roots) $p \ll q$ or $q \ll p$.
2. (Hermite-Kakeya-Obreschkoff) $ap + bq$ is real-rooted for all $a, b \in \mathbb{R}$.
3. (Hermite-Biehler) $p + iq$ is either $\mathcal{H}_+$-stable or $\mathcal{H}_-$-stable.

Fact: This extends to multivariate stable and real stable polynomials.

We write $p \ll q$ if $p(\boldsymbol{a} \cdot t + \boldsymbol{b}) \ll q(\boldsymbol{a} \cdot t + \boldsymbol{b})$ for all $\boldsymbol{a} \in \mathbb{R}_+^n$ and $\boldsymbol{b} \in \mathbb{R}^n$.
Some interlacing property of the real varieties.

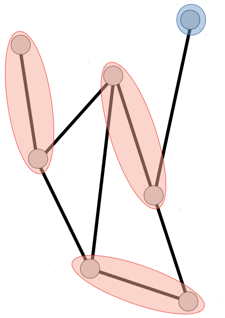
Idea: T preserves real stability and $p + iq$ stable $\implies$
$T[ap + bq] = aT[p] + bT[q]$ real stable for all $a, b \in \mathbb{R} \implies$
$T[p] + iT[q] = T[p + iq]$ is $\mathcal{H}_+^n$-stable or $\mathcal{H}_-^n$-stable.

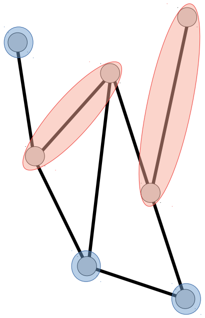
Now: One-dimension condition, plus HKO $\implies$ two-dimension condition.

Outline

Univariate matching polynomial

Given a graph $G = (V, E)$, a k-**matching** M of G is a selection of k edges for which no two edges touch the same vertex.





Univariate matching polynomial: $M_G(t) := \sum_{k=0}^{\lfloor |V|/2 \rfloor} (-1)^k m_k t^{|V|-2k}.$

Multivariate matching polynomial

Multivariate matching polynomial:

$$\mu_G(\boldsymbol{x}) = \mu_G((x_v)_{v \in V}) := \sum_M \prod_{\substack{u \sim v \\ u,v \in M}} -x_u x_v \in \mathbb{R}^{\mathbf{1}}[\boldsymbol{x}].$$

How does this relate to the univariate polynomial?

$$\sum_{k=0}^{\lfloor |V|/2 \rfloor} (-1)^k m_k t^{|V|-2k} = M_G(t) = t^{|V|} \cdot \mu_G(t^{-1}, \ldots, t^{-1}).$$

Generalization of Heilmann-Lieb theorem: μ_G is real stable. **True?**

The polynomial μ_G much more naturally corresponds to the graph G.
Geometry of polynomials principle: More variables make things easier.

Key idea: We can exploit this to get much simpler expressions for μ_G than for M_G (recall the deletion-contraction recursion for M_G).

Expressing the multivariate matching polynomial

First consider:

$$p_G(\boldsymbol{x}) := \prod_{(u,v)\in E} (1 - x_u x_v) = \sum_{S\subseteq E} \prod_{(u,v)\in S} -x_u x_v.$$

This is a real stable polynomial. **How does this relate to matchings?**

$S \subseteq E$ corresponds to a matching iff $\prod_{(u,v)\in S} -x_u x_v$ is multiaffine.

Why? k edges incident on $v \in V$ yields x_v^k in the term associated to S.

Next question: How does the following linear operator relate to stability?

$$\text{MAP} : \mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}] \to \mathbb{R}^{\boldsymbol{1}}[\boldsymbol{x}] \quad \text{via} \quad \text{MAP} : \boldsymbol{x}^{\boldsymbol{\mu}} \mapsto \begin{cases} \boldsymbol{x}^{\boldsymbol{\mu}} & \forall i,\ \mu_i \in \{0,1\} \\ 0 & \exists i,\ \mu_i \geq 2 \end{cases}.$$

MAP := "MultiAffine Part" and $\mu_G = \text{MAP}(p_G)$.

The multiaffine part (MAP) operator

Definition of MAP:

$$\text{MAP} : \mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}] \to \mathbb{R}^{\mathbf{1}}[\boldsymbol{x}] \quad \text{via} \quad \text{MAP} : \boldsymbol{x}^{\boldsymbol{\mu}} \mapsto \begin{cases} \boldsymbol{x}^{\boldsymbol{\mu}} & \forall i,\ \mu_i \in \{0,1\} \\ 0 & \exists i,\ \mu_i \geq 2 \end{cases}.$$

Symbol of MAP (here $\boldsymbol{x}^S := \prod_{i \in S} x_i$):

$$\text{MAP}\left[\prod_{i=1}^{n} (x_i + z_i)^{\lambda_i}\right] = \sum_{\boldsymbol{\mu} \leq \boldsymbol{\lambda}} \binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}} \boldsymbol{z}^{\boldsymbol{\lambda} - \boldsymbol{\mu}} \cdot \text{MAP}[\boldsymbol{x}^{\boldsymbol{\mu}}] = \boldsymbol{z}^{\boldsymbol{\lambda}} \sum_{\boldsymbol{\mu} \leq \mathbf{1}} \frac{\boldsymbol{\lambda}^{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}}{\boldsymbol{z}^{\boldsymbol{\mu}}}$$

$$= \boldsymbol{z}^{\boldsymbol{\lambda}} \prod_{i=1}^{n} \left(\frac{\lambda_i x_i}{z_i} + 1\right) = \boldsymbol{z}^{\boldsymbol{\lambda} - \mathbf{1}} \prod_{i=1}^{n} (\lambda_i x_i + z_i).$$

This is a real stable polynomial. $\implies$ MAP **preserves real stability.**

Corollary: p_G is real stable $\implies$ $\mu_G = \text{MAP}(p_G)$ is real stable. $\implies$ **Strengthening** of the Heilmann-Lieb theorem with straightforward proof!

Another proof of real stability

Consider the following polynomial, given a graph $G = (V, E)$:

$$p_G(\boldsymbol{x}) := \prod_{(u,v)\in E} (1 - \partial_{x_u}\partial_{x_v}) \prod_{v\in V} x_v = \sum_{S\subseteq E} \prod_{(u,v)\in S} (-\partial_{x_u}\partial_{x_v}) \prod_{v\in V} x_v \in \mathbb{R}^{\mathbf{1}}[\boldsymbol{x}].$$

If $S \subseteq E$ contains k edges incident on the same vertex u, then $\partial_{x_u}^k$ appears. This maps $\prod_{v\in V} x_v$ to 0 for $k \geq 2$. **Therefore:** The only non-zero terms correspond to matchings of G:

$$p_G(\boldsymbol{x}) = \sum_M \prod_{(u,v)\in M} (-\partial_{x_u}\partial_{x_v}) \prod_{v\in V} x_v = \boldsymbol{x}^{\mathbf{1}} \cdot \mu_G(x_{v_1}^{-1}, \ldots, x_{v_n}^{-1}).$$

Now: $\prod_{v\in V} x_v$ is real stable. **To show:** $1 - \partial_{x_u}\partial_{x_v}$ preserves real stability.

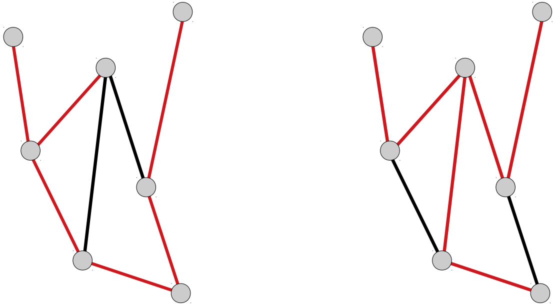
$$(1 - \partial_{x_j}\partial_{x_k}) \left[\prod_{i=1}^n (x_i + z_i)\right] = [(x_j + z_j)(x_k + z_k) - 1] \cdot \prod_{i\neq j,k} (x_i + z_i).$$

Finally: $x_j, z_j, x_k, z_k \in \mathcal{H}_+ \implies (x_j + z_j)(x_k + z_k) \neq 1.$

Outline

Spanning trees of a graph

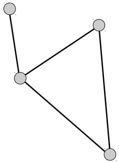
Given a connected graph $G = (V, E)$, a **spanning tree** T of G is a selection of $|V| - 1$ edges which touch every vertex and contain no cycles.



Spanning tree polynomial: $T_G(\boldsymbol{x}) := \sum_T \boldsymbol{x}^T = \sum_T \prod_{e \in T} x_e \in \mathbb{R}^{\mathbf{1}}[\boldsymbol{x}]$.

Matrix tree theorem

Graph Laplacian: [diagonal matrix of degrees] − [incidence matrix]



$$\implies L_G = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \implies L_G \cdot \mathbf{1} = 0.$$

Kirchoff: Any size $|V| - 1$ principal minor counts spanning trees.

$$\det \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} = 1 \cdot (6-1) - (-1) \cdot (-2-0) + 0 \cdot (1-0) = 3.$$

How to generalize? Want an edge-weighted count, via variables. **First:**

$$L_G := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}^\top \Longleftarrow \text{"edge matrix".}$$

Generalized matrix tree theorem

Now add variables:

$$L_G(\boldsymbol{x}) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \cdot \operatorname{diag}(\boldsymbol{x}) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}^\top .$$

Setting $x_e \in \{0,1\}$ gives L_H for a subgraph H with specified edges. Then, any size $|V|-1$ principal minor counts spanning trees in the subgraph.

Proposition: $T_G(\boldsymbol{x}) = p(\boldsymbol{x}) :=$ any size $|V|-1$ principal minor of $L_G(\boldsymbol{x})$.
Proof: We have $2^{|E|}$ evaluations and $2^{|E|}$ coefficients. The coefficients can be computed explicitly via induction:

$$p_{\mathbf{0}} = p(\mathbf{0}),\ p_{\boldsymbol{e}_k} = p(\boldsymbol{e}_k) - p(\mathbf{0}),\ p_{\boldsymbol{e}_j+\boldsymbol{e}_k} = p(\boldsymbol{e}_j+\boldsymbol{e}_k) - p(\boldsymbol{e}_j) - p(\boldsymbol{e}_k) + p(\mathbf{0}),\ \ldots$$

This shows that all coefficients are computable via these evaluations. Since T_G and p agree on these evaluations, they must be equal.

Real stability of the spanning tree polynomial

Therefore: $T_G(\boldsymbol{x}) = \det(A \cdot \mathrm{diag}(\boldsymbol{x}) \cdot A^\top)$ for some real matrix A.

Corollary: $T_G(\boldsymbol{x})$ is a real stable polynomial.

Proof: For any $\boldsymbol{x} = \boldsymbol{a} + i \cdot \boldsymbol{b} \in \mathcal{H}_+^n$, we have

$$T_G(\boldsymbol{x}) = \det(A \cdot \mathrm{diag}(\boldsymbol{a} + i \cdot \boldsymbol{b}) \cdot A^\top) = \det(H + i \cdot P),$$

where H is Hermitian and P is positive definite (or else $T_G(\boldsymbol{b}) = 0 \equiv T_G$).

Now for any $\boldsymbol{w} \in \mathbb{C}^n$, we have

$$\boldsymbol{w}^*(H + i \cdot P)\boldsymbol{w} = r + i \cdot p,$$

where $r \in \mathbb{R}$ and $p > 0$.

Therefore $H + i \cdot P$ is nonsingular, and this completes the proof.

Fact: $p|_{x_e=0}$ = deletion, ∂_{x_e} = contraction (stability preservers).

The strong Rayleigh (SR) conditions revisited

The (multiaffine) spanning tree polynomial is real stable:

$$T_G(\boldsymbol{x}) = \sum_T \prod_{e \in T} x_e \in \mathbb{R}^{\mathbf{1}}[\boldsymbol{x}].$$

Multiaffine $\implies$ strong Rayleigh conditions. **What do they mean?**
First: Let's just evaluate the Rayleigh expression R_{ef} at $\mathbf{1}$:

$$\begin{aligned} R_{ef}[T_G](\mathbf{1}) &= \partial_{x_e} T_G \cdot \partial_{x_f} T_G - T_G \cdot \partial_{x_e} \partial_{x_f} T_G \big|_{\boldsymbol{x}=\mathbf{1}} \\ &= \#(e \in T) \cdot \#(f \in T) - \#(T) \cdot \#(e, f \in T) \geq 0. \end{aligned}$$

Equivalent: Divide through by $\#(T)^2$ and rearrange to get:

$$\mathbb{P}[e \in T] \geq \frac{\mathbb{P}[e, f \in T]}{\mathbb{P}[f \in T]} = \mathbb{P}[e \in T \mid f \in T].$$

"Negative correlation". Other evaluations give the same conclusion for edge-weighted probability distributions on the spanning trees of G.

Foreshadowing: Matroid basis-generating polynomials

The spanning trees of G = **set of bases of a graphic matroid**.

Matroid: $M = (E, \mathcal{I})$ where E is the **ground set** and $\mathcal{I} \subseteq 2^E$ are the **independent** subsets, which satisfy:

1. **Nonempty:** $\mathcal{I} \neq \varnothing$.
2. **Hereditary:** $B \in \mathcal{I}$ and $A \subseteq B$ implies $A \in \mathcal{I}$.
3. **Exchange/Augmentation:** For all $A, B \in \mathcal{I}$ such that $|A| < |B|$, there exists $e \in B \setminus A$ such that $A \cup \{e\} \in \mathcal{I}$.

E.g.: A set of vectors in a vector space, with $\mathcal{I}$ given by linearly independent subsets (**linear matroid**). The set of edges of a graph, with $\mathcal{I}$ given by subsets with no cycles (**graphic matroid**). **Many more...**

Maximal $B \in \mathcal{I}$ are the **bases**, $\mathcal{B} \subset \mathcal{I}$, of M. **Another definition of M:**

3. **Exchange:** For any bases $B_1, B_2 \in \mathcal{B}$ and any $e_1 \in B_1 \setminus B_2$, there exists $e_2 \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}$.

The spanning tree polynomial is a basis-generating polynomial. **Others?**

Outline

Rayleigh conditions

Usually only care about strong Rayleigh conditions evaluated in the positive orthant, for homogeneous multiaffine polynomials. Or even just evaluated at $\boldsymbol{x} = \mathbf{1}$.

Strong Rayleigh is stronger: allows all real evaluations.

Open question: Is there a property which implies Rayleigh conditions in the positive orthant (which is weaker than real stability), which has a nice theory of linear preservers (something like the BB characterization)?

Next week: Lorentzian (aka strongly/completely log-concave) polynomials have a Rayleigh-type property, but it is slightly too weak.

Application: Random cluster model in the "fewer clusters" regime ($q < 1$).

Unitary BB characterization

BB characterization: T preserves real stability if $T\left[\prod_{i=1}^{n}(x_i z_i + 1)^{\lambda_k}\right]$ is real stable. This is related to an $\mathrm{SU}_2(\mathbb{C})$-invariant bilinear form.

Unitary version for $\mathrm{SU}_n(\mathbb{C})$: Let T be a map between spaces of homogeneous polynomials $\mathbb{R}_h^d[x_1 : \cdots : x_n]$ of **total** degree d. Consider a different symbol of T:

$$\mathrm{Symb}[T](x_1 : \cdots : x_n, z_1 : \cdots : z_n) := T\left[\left(\sum_{i=1}^{n} x_i z_i\right)^d\right].$$

Open question: Is there a BB-like characterization for some nice class of polynomials using this symbol?

Really want some class which generalizes real stable polynomials. This could imply Gurvits's capacity conjecture. **Maybe Rayleigh conditions?**