Multivariate Polynomials

Polynomial Capacity: Theory, Applications, Generalizations

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Notation

"Usual polynomials":

- $\mathbb{C}[x]$:= v.s. of complex polynomials in one variable.
- $\mathbb{C}^d[x]$:= v.s. of polynomials of degree at most d.
- For $p \in \mathbb{C}^d[x]$, we write $p(x) = \sum_{k=0}^d p_k x^k$.
- monic := the leading coeffcient is 1.
- $\deg(p)$:= the degree of the polynomial.
- $\lambda(p)$:= the roots/zeros of the polynomial, counting multiplicity.
- $\frac{d}{dx} = \frac{\partial}{\partial x} = \partial_x$:= derivative with respect to x.

Polynomials with zeros in projective space:

- $\mathbb{C}_h^d[x : y]$:= v.s. of bivariate homogeneous polynomials of degree d.
- For $p \in \mathbb{C}_h^d[x : y]$, we write $p(x : y) = \sum_{k=0}^d p_k x^k y^{d-k}$.
- $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$:= complex projective line, or Riemann sphere.
- $\mathrm{SL}_2(\mathbb{C})$:= 2×2 invertible complex matrices, $\det = 1$.
- ∂_y := derivative with respect to y.

Recall: The big three

The **geometry of polynomials** is generally an investigation of the connections between the various properties of polynomials:

- **Algebraic**, via the roots/zeros of the polynomial.
- **Combinatorial**, via the coefficients of the polynomial.
- **Analytic**, via the evaluations of the polynomial.

Why do we care? We use features of the interplay between these three to prove facts about mathematical objects which a priori have nothing to do with polynomials.

Typical method:

1. Encode some object as a polynomial which has some nice properties.
2. **Apply operations** to that polynomial which preserve those properties.
3. Extract information at the end which relates back to the object.

Outline

Outline

Laguerre's theorem

Recall Laguerre's theorem: If

1. $C \subset \mathbb{CP}^1$ is a circular region (disc, half-plane, complement of disc),
2. $p \in \mathbb{C}_h^d[x:y]$ has all its roots in C,
3. $(a:b) \in \mathbb{CP}^1$ is not in C,

then $(a\partial_x + b\partial_y)p$ has all its roots in C.

"Usual polynomials" version: If

1. $C \subset \mathbb{C}$ is a circular region,
2. $p \in \mathbb{C}^d[x]$ has all its roots in C ($d - \deg(p)$ counts roots at ∞),
3. $a \in \mathbb{C}$ is not in C,

then $(a\partial_x + \partial_y)p := d \cdot p(x) - (x-a) \cdot p'(x)$ has all its roots in C.

Idea: $a\partial_x + b\partial_y = \phi^{-1} \circ \partial_x \circ \phi =$ rotate roots in $\mathbb{CP}^1$ via $\phi \in \mathrm{SL}_2(\mathbb{C})$, then take derivative and apply Gauss-Lucas, then rotate roots back.
Also: $(a\partial_x + b\partial_y)p \not\equiv 0$, or else implies $p(a:b) = 0$.

Grace's theorem

First: If $(r_i : s_i) \notin C$ and $(a_1 : b_1) \in C$, then

$$f(x : y) = (a_1 \partial_x + b_1 \partial_y) \prod_{i=1}^{d} (s_i x - r_i y) \in \mathbb{C}_h^{d-1}[x : y]$$

has all roots outside of C.

Next: If $(r_i : s_i) \notin C$ and $(a_i : b_i) \in C$, then

$$f(x : y) = \prod_{i=1}^{d} (a_i \partial_x + b_i \partial_y) \prod_{i=1}^{d} (s_i x - r_i y) \in \mathbb{C}_h^{0}[x : y]$$

has all roots outside of C. $\implies f \not\equiv 0$.

Grace's theorem: If $p, q \in \mathbb{C}_h^d[x : y]$ are such that p has all roots in C and q has no roots in C, then $\langle p, q \rangle^d := p(\partial_y : -\partial_x) q(x : y) \neq 0$.

Aside: $\langle p, q \rangle^d = D^d(p(x : y) q(z : w))$ for $D = \partial_x \partial_w - \partial_y \partial_z$, where D is $\mathrm{SL}_2(\mathbb{C})$-invariant $\implies \langle \cdot, \cdot \rangle^d$ is unique $\mathrm{SL}_2(\mathbb{C})$-invariant bilinear form.

Grace's theorem: Why do we care?

Bilinear form $\langle \cdot, \cdot \rangle^d$ is non-zero when roots are separated. **So what?**

Many classical theorems are proven using Grace's theorem. **How?**

First: We can interpret the bilinear form as a choice of isomorphism between $\mathbb{C}_h^d[x:y]$ and its dual space $\mathbb{C}_h^d[x:y]^*$ via $p \longleftrightarrow \langle p, \cdot \rangle^d$.

Next: Induce a map from linear operators on polynomials to polynomials in more variables. Letting $\mathcal{L}_d$ denote the space of operators,

$$\mathcal{L}_d \cong \mathbb{C}_h^d[x:y] \otimes \mathbb{C}_h^d[x:y]^* \overset{\downarrow}{\cong} \mathbb{C}_h^d[x:y] \otimes \mathbb{C}_h^d[x:y] \cong \mathbb{C}_h^{(d,d)}[x:y, z:w].$$

Denoting this by $\mathrm{Symb}^d : \mathcal{L}_d \xrightarrow{\sim} \mathbb{C}_h^{(d,d)}[x:y, z:w]$ gives:

$$T[p](x:y) = \left\langle \mathrm{Symb}^d[T](x:y, z:w), p(z:w) \right\rangle^d.$$

Finally: Zero location of p and $\mathrm{Symb}^d[T]$ implies non-vanishing of $T[p]$.

Multivariate polynomials?

Now: Multivariate theory

- The idea of Symb^d requires multivariate polynomials.
- Multivariate allows more direct connection to mathematical objects.
- **E.g.:** Matching polynomial with one variable per vertex.
- **Problem:** No analogue to the fundamental theorem of algebra.
- **Problem:** Zeros cannot be contained to some compact region.

Hint: Given $p \in \mathbb{C}_h^d[x : y]$ with roots $(r_i : s_i) \notin C$, define

$$P(z_1 : w_1, \ldots, z_d : w_d) := \frac{1}{d!} \prod_{i=1}^{d} (z_i \partial_x + w_i \partial_y) \prod_{i=1}^{d} (s_i x - r_i y).$$

P is a multivariate polynomial of degree one in each variable $(z_i : w_i)$.

Grace's theorem: P has no zeros in $C \times \cdots \times C$.

Walsh coincidence theorem

The **polarization** of a polynomial p:

$$P(z_1 : w_1, \ldots, z_d : w_d) := \frac{1}{d!} \prod_{i=1}^{d} (z_i \partial_x + w_i \partial_y) \prod_{i=1}^{d} (s_i x - r_i y).$$

Properties:

- P is symmetric.
- P is of degree one in each variable (**multiaffine**).
- $P(z : w, \ldots, z : w) = p(z : w)$, via product rule:

$$(z\partial_x + w\partial_y) \prod_{i=1}^{d} (s_i x - r_i y) = \sum_{i=1}^{d} (s_i z - r_i w) \prod_{j \neq i} (s_j x - r_j y) = \cdots$$

- The **unique** polynomial with these properties.

Walsh coincidence theorem: A polynomial p has no roots in C if and only if its **polarization** has no roots in C^d.

Multiaffine equivalence

Walsh coincidence theorem for "usual polynomials":

$$p(x) = \sum_{k=0}^{d} \binom{d}{k} \tilde{p}_k x^k \quad \iff \quad P(\boldsymbol{x}) = \sum_{S \in \binom{[d]}{k}} \tilde{p}_{|S|} \boldsymbol{x}^S = \sum_{k=0}^{d} \sum_{|S|=k} \tilde{p}_k \boldsymbol{x}^S$$

Stability in a circular region is preserved (stability = no zeros).

Stability-preserving map between $\mathbb{C}^d[x]$ and $\mathbb{C}^{(1,\ldots,1)}[x_1, \ldots, x_d]$ (or equivalently between $\mathbb{C}_h^d[x : y]$ and $\mathbb{C}_h^{(\mathbf{1}^d)}[\boldsymbol{x} : \boldsymbol{y}]$).

More variables: $\mathbb{C}^{(\lambda_1,\ldots,\lambda_n)}[x_1, \ldots, x_n] \iff \mathbb{C}^{(\mathbf{1}^{\lambda_1},\ldots,\mathbf{1}^{\lambda_n})}[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$.

Conceptual takeaway: Stability properties for a space of polynomials equivalent to properties on some isomorphic symmetric multiaffine space.

Outline

Real stability

Multivariate polynomial spaces: $\mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}] = \mathbb{C}^{(\lambda_1,\ldots,\lambda_n)}[x_1,\ldots,x_n]$ = set of all polynomials in variables $x_1,\ldots,x_n$ of degree at most λ_i in x_i for all i.

Note: From now on, mainly work with "usual polynomial" spaces, but remember that everything is equivalent.

Very important definition: A polynomial $p \in \mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ is **stable** if:

$$x_1,\ldots,x_n \in \mathcal{H}_+ = \text{upper half-plane} \implies p(x_1,\ldots,x_n) \neq 0.$$

If we further have $p \in \mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$, then p is **real stable**.

E.g.: Matching poly., spanning tree poly., $\det\left(\sum_i A_i x_i\right)$ for A_i PSD.

Walsh coincidence theorem says p is stable iff the polarization P is.

Can always keep in mind any other circular region C and use C^n-**stable**.
E.g.: $\mathcal{H}_+^n$-stable and $\mathcal{H}_-^n$-stable for upper and lower half-plane stability.

Real stable: the "right" generalization of real-rooted

Fact: If $p \in \mathbb{R}^d[x]$ is univariate, then p is real stable iff p is real-rooted.
Proof: For real polynomials, non-real zeros come in conjugate pairs.

Equivalent properties for $p \in \mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$:

1. **Zero location:** p is real stable.
2. **Linear restrictions:** $p(\boldsymbol{a}t + \boldsymbol{b}) \in \mathbb{R}^{\lambda_1 + \cdots + \lambda_n}[t]$ is real-rooted for all $\boldsymbol{a} \in \mathbb{R}^n_+$ (positive orthant) and $\boldsymbol{b} \in \mathbb{R}^n$.
3. **Strong Rayleigh [Brändén '07]:** For p multiaffine ($\boldsymbol{\lambda} = \mathbf{1}$),

$$R_{ij}(p) := \left(\partial_{x_i} p \cdot \partial_{x_j} p - p \cdot \partial_{x_i}\partial_{x_j} p\right)\Big|_{x_i = x_j = 0} \geq 0 \quad \text{for } x \in \mathbb{R}^n, \text{ all } i, j.$$

Example: $p(x_1, x_2) = x_1 x_2 - 1$.

1. $x_1, x_2 \in \mathcal{H}_+ \implies x_1 x_2 \neq 1 \implies x_1 x_2 - 1 \neq 0$
2. $(a_1 t + b_1)(a_2 t + b_2) - 1 = a_1 a_2 t^2 + (a_1 b_2 + a_2 b_1)t + (b_1 b_2 - 1)$
$\implies (a_1 b_2 + a_2 b_1)^2 - 4 a_1 a_2 (b_1 b_2 - 1) = (a_1 b_2 - a_2 b_1)^2 + 4 a_1 a_2 \geq 0$
3. $i = 1, j = 2 \implies x_2 \cdot x_1 - (x_1 x_2 - 1) \cdot 1 = 1 \geq 0$

An aside on the strong Rayleigh conditions

For $p \in \mathbb{R}^{\mathbf{1}}[\boldsymbol{x} : \boldsymbol{y}] \cong \mathbb{R}^{\mathbf{1}}[\boldsymbol{x}]$, there are p_0, p_1 independent of x_1, y_1 such that:

$$p = x_1 \cdot p_1 + y_1 \cdot p_0 \in \mathbb{R}^{\mathbf{1}}[\boldsymbol{x} : \boldsymbol{y}] \iff q = x_1 \cdot p_1 + p_0 \in \mathbb{R}^{\mathbf{1}}[\boldsymbol{x}]$$

This means: $\partial_{y_1} p$ on $\mathbb{R}^{\mathbf{1}}[\boldsymbol{x} : \boldsymbol{y}]$ is equivalent to $q\big|_{x_1=0}$ on $\mathbb{R}^{\mathbf{1}}[\boldsymbol{x}]$.

So: $R_{ij}(p) = \partial_{x_i}\partial_{y_j} p \cdot \partial_{y_i}\partial_{x_j} p - \partial_{x_i}\partial_{x_j} p \cdot \partial_{y_i}\partial_{y_j} p$ for $p \in \mathbb{R}^{\mathbf{1}}[\boldsymbol{x} : \boldsymbol{y}]$.
$\implies$ R_{ij} is $\mathrm{SL}_2(\mathbb{C})$-invariant. (Recall the D map from Grace's theorem.)

Therefore: Version of strong Rayleigh for all circular regions.

Open question: Strong Rayleigh is crucial to capacity bounds. Can one get similar bounds for other circular regions?

Proof of equivalence

1. **Zero location:** p is real stable.
2. **Linear restrictions:** $p(\boldsymbol{a}t + \boldsymbol{b}) \in \mathbb{R}^{\lambda_1+\cdots+\lambda_n}[t]$ is real-rooted for all $\boldsymbol{a} \in \mathbb{R}^n_+$ (positive orthant) and $\boldsymbol{b} \in \mathbb{R}^n$.
3. **Strong Rayleigh [Brändén '07]:** For p multiaffine ($\boldsymbol{\lambda} = \mathbf{1}$),

$$R_{ij}(p) := \left(\partial_{x_i} p \cdot \partial_{x_j} p - p \cdot \partial_{x_i}\partial_{x_j} p\right)\Big|_{x_i=x_j=0} \geq 0 \quad \text{for } x \in \mathbb{R}^n, \text{ all } i,j.$$

(1) $\implies$ (2): If $p(\boldsymbol{a}(\beta + \alpha i) + \boldsymbol{b}) = 0$ for $\alpha, \boldsymbol{a} > 0$, then p is not stable.
(2) $\implies$ (1): Follows since $p(\boldsymbol{a} \cdot i + \boldsymbol{b}) \neq 0$ for all $\boldsymbol{a} > 0$.

(3) $\implies$ (1): Exercise (we won't really need this direction).
(1) $\implies$ (3): By evaluating all variables except x_i, x_j, we can reduce to $p = ax_ix_j + bx_i + cx_j + d$. Evaluation preserves real stability (or $\equiv 0$) by evaluating $r + \epsilon i$ and limiting $\epsilon \to 0^+$. We then have

$$R_{ij}(p) = \partial_{x_i} p \cdot \partial_{x_j} p - p \cdot \partial_{x_i}\partial_{x_j} p\Big|_{x_i=x_j=0} = bc - ad.$$

Proof equivalence (continued)

For $p = ax_i x_j + bx_i + cx_j + d$, need to show $bc \geq ad$.

Case 1: $a \neq 0$. First scale the whole polynomial so that $a = 1$. Now compute:

$$p(t-c, t-b) = (t-c)(t-b) + b(t-c) + c(t-b) + d = t^2 + d - bc.$$

By property (2), this is real-rooted $\implies d - bc \leq 0$ via discriminant.

Case 2: $a = 0$. Need to show $bc \geq 0$. If not, then $bc < 0$ and we have:

$$p(|c| \cdot i - b^{-1}d, |b| \cdot i) = (b|c| + c|b|)i + (d - d) = 0.$$

This contradicts the stability of p.

Outline

1. Grace's theorem and corollaries
   - Grace's theorem
   - Multiaffine polynomials and the Walsh coincidence theorem
2. Stability and multivariate polynomials
   - Real stability
   - The strong Rayleigh conditions
   - Proof of equivalence
3. Stability preservers
   - The Borcea-Brändén characterization
   - The BB characterization for real stability
   - The less important direction of the proof
4. Open problems

Stability preservers

Gauss-Lucas theorem: ∂_x preserves stability for univariate polynomials.

Corollary: ∂_{x_j} preserves (real) stability for multivariate polynomials.
Proof: WLOG, $j = n$. Let $q(t) := p(a_1 i + b_1, \ldots, a_{n-1} i + b_{n-1}, t)$ for fixed $\boldsymbol{a} \in \mathbb{R}^n_+$ and $\boldsymbol{b} \in \mathbb{R}^n$, which is stable. We then have:

$$\partial_{x_n} p(a_1 i + b_1, \ldots, a_n i + b_n) = \partial_t q(a_n i + b_n) \neq 0.$$

Corollary: Positive orthant directional derivatives preserve (real) stability.
Proof: We want to prove that $\boldsymbol{a} \cdot \nabla p$ is stable for $\boldsymbol{a} \in \mathbb{R}^n_+$. Note first that $f(x_1, \ldots, x_n, t) := p(\boldsymbol{x} + \boldsymbol{a} \cdot t)$ is a stable polynomial, since $\boldsymbol{x} + \boldsymbol{a} \cdot t \in \mathcal{H}_+$ for $x_j, t \in \mathcal{H}_+$ and $\boldsymbol{a} \in \mathbb{R}^n_+$. So,

$$\partial_t f\big|_{t=0} = \boldsymbol{a} \cdot \nabla p(\boldsymbol{x} + \boldsymbol{a} \cdot 0) = \boldsymbol{a} \cdot \nabla p$$

is also stable.

What about other linear operators?

The Borcea-Brändén characterization (stable)

Definition: The **symbol** of a linear operator $T : \mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}] \to \mathbb{C}[\boldsymbol{x}]$:

$$\mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x}, \boldsymbol{z}) := T\left[\prod_{i=1}^{n} (x_i + z_i)^{\lambda_i}\right] = \sum_{\boldsymbol{0} \leq \boldsymbol{\mu} \leq \boldsymbol{\lambda}} \binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}} \boldsymbol{z}^{\boldsymbol{\lambda}-\boldsymbol{\mu}} T[\boldsymbol{x}^{\boldsymbol{\mu}}].$$

Here T acts only on $\boldsymbol{x}$, $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ is entrywise, and $\binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}} := \prod_i \binom{\lambda_i}{\mu_i}$.

Theorem (Borcea-Brändén '09)

For a given linear operator $T : \mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}] \to \mathbb{C}[\boldsymbol{x}]$, we have that T preserves stability (allowing $\equiv 0$) if and only if one of the following holds:

1. $\mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x}, \boldsymbol{z})$ *is stable.*
2. *The image of T is a one-dimensional space of stable polynomials.*

Conceptual takeaway: T preserves stability "iff" its symbol is stable.

Proof of the stable BB characterization

Most important: $\text{Symb}^\lambda[T](\boldsymbol{x}, \boldsymbol{z})$ is stable implies T preserves stability. (We'll come back to the other direction.)

Recall: Multiaffine stability equivalent to more general cases by Walsh coincidence theorem. **So let's prove it for multiaffine first.**

Proof: Up to positive scalar, for any $p \in \mathbb{C}^{\mathbf{1}}[\boldsymbol{t}]$ we have

$$T[p](\boldsymbol{x}) = \prod_{i=1}^{n} (\partial_{z_i} + \partial_{t_i}) \Big|_{\boldsymbol{z}=\boldsymbol{t}=0} \left[\text{Symb}^{\mathbf{1}}[T](\boldsymbol{x}, \boldsymbol{z}) \cdot p(\boldsymbol{t}) \right].$$

Why? $\prod_{i=1}^{n} (\partial_{z_i} + \partial_{t_i})|_{\boldsymbol{z}=\boldsymbol{t}=0} [f(\boldsymbol{z}) \cdot g(\boldsymbol{t})]$ is a bilinear form on polynomials. $\text{Symb}^{\mathbf{1}}$ **is then the corresponding induced map between linear operators and polynomials**, as discussed above.

Since $\text{Symb}^{\mathbf{1}}[T](\boldsymbol{x}, \boldsymbol{z}) \cdot p(\boldsymbol{t})$ is stable, and since $(\partial_{z_i} + \partial_{t_i})$ and evaluating variables at 0 both preserve stability, we have that $T[p](\boldsymbol{x})$ is also stable.

More general polynomials

Proof is done for multiaffine polynomials. **What about higher degree?**

Use polarization: $p \in \mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ is stable iff $\mathrm{Pol}[p] \in \mathbb{C}^{\boldsymbol{1}}[\boldsymbol{x}]$ is.

Last piece: How does polarization relate to Symb? **Easy answer!**

$$\begin{aligned}
\mathrm{Pol}\left[\mathrm{Symb}^{\boldsymbol{\lambda}}[T]\right] &= (\mathrm{Pol} \circ T)\left[\prod_{i=1}^{n} (x_i + z_i)^{\lambda_i}\right] \\
&= \left(\mathrm{Pol} \circ T \circ \mathrm{Pol}^{-1}\right)\left[\prod_{i,j} (x_{i,j} + z_{i,j})\right] \\
&= \mathrm{Symb}^{\boldsymbol{1}}\left[\mathrm{Pol} \circ T \circ \mathrm{Pol}^{-1}\right].
\end{aligned}$$

Finally: $\mathrm{Symb}^{\boldsymbol{\lambda}}[T]$ is stable iff $\mathrm{Pol}\left[\mathrm{Symb}^{\boldsymbol{\lambda}}[T]\right] = \mathrm{Symb}^{\boldsymbol{1}}\left[\mathrm{Pol} \circ T \circ \mathrm{Pol}^{-1}\right]$ is stable iff $\left(\mathrm{Pol} \circ T \circ \mathrm{Pol}^{-1}\right)$ preserves stability iff T preserves stability.

The BB characterization (real stable)

Theorem (Borcea-Brändén '09)

For a given linear operator $T : \mathbb{R}^{\lambda}[\mathbf{x}] \to \mathbb{R}[\mathbf{x}]$, we have that T preserves real stability (allowing $\equiv 0$) if and only if one of the following holds:

1. $\mathrm{Symb}^{\lambda}[T](\mathbf{x}, \mathbf{z})$ *is real stable.*
2. $\mathrm{Symb}^{\lambda}[T](\mathbf{x}, -\mathbf{z})$ *is real stable.*
3. *The image of T is a two-dimensional space of real stable polynomials.*

Two conditions now? Real stable iff $\mathcal{H}_+^n$-stable iff $\mathcal{H}_-^n$-stable.

1. $\mathrm{Symb}^{\lambda}[T](\mathbf{x}, \mathbf{z})$: preserves (real) stability by previous theorem.
2. $\mathrm{Symb}^{\lambda}[T](\mathbf{x}, -\mathbf{z})$: maps $\mathcal{H}_+^n$-stable to $\mathcal{H}_-^n$-stable by prev. theorem.

Another option: $\mathrm{Symb}^{\lambda}[T](\mathbf{x} \cdot \mathbf{z}, \mathbf{1}) = \mathbf{z}^{\lambda} \cdot \mathrm{Symb}^{\lambda}[T](\mathbf{x}, \mathbf{z}^{-1})$ where the product $\mathbf{x} \cdot \mathbf{z}$ is entrywise. **This one is useful for capacity.**

The other direction of the BB characterization

For the complex case: Suppose $\mathrm{Symb}^{\mathbf{1}}[T](\boldsymbol{x}, \boldsymbol{z})$ is not stable. Recall:

$$T[p](\boldsymbol{x}) = \prod_{i=1}^{n} (\partial_{z_i} + \partial_{t_i}) \left[\mathrm{Symb}^{\mathbf{1}}[T](\boldsymbol{x}, \boldsymbol{z}) \cdot p(\boldsymbol{t})\right].$$

Pick $\boldsymbol{x}, \boldsymbol{\alpha} \in \mathcal{H}_+^n$ such that $\mathrm{Symb}^{\mathbf{1}}[T](\boldsymbol{x}, \boldsymbol{\alpha}) = 0$, and define the $\mathcal{H}_+^n$-stable polynomial $p(\boldsymbol{t}) := \prod_{i=1}^n (t_i + \alpha_i)$ to get:

$$T[p](\boldsymbol{x}) = \prod_{i=1}^{n} (\partial_{z_i} + \partial_{t_i}) \left[\mathrm{Symb}^{\mathbf{1}}[T](\boldsymbol{x}, \boldsymbol{z}) \cdot p(\boldsymbol{t})\right] = \mathrm{Symb}^{\mathbf{1}}[T](\boldsymbol{x}, \boldsymbol{\alpha}) = 0.$$

So either T does not preserve stability or $T[p] \equiv 0$. If T preserves stability, then $T[B_\epsilon(p)]$ is an open set containing 0. By scaling, this implies the whole image of T consists of stable polynomials.

Fact: A vector space of stable polynomials is of dimension at most 1.
Proof: Exercise. Is there some algebraic geometry way to see this?

The other direction of the BB characterization

For the real case: Need a link between real stability and complex stability.

Recall: In the univariate case, the following are equivalent.

1. (Interlacing roots) $p \ll q$ or $q \ll p$.
2. (Hermite-Kakeya-Obreschkoff) $ap + bq$ is real-rooted for all $a, b \in \mathbb{R}$.
3. (Hermite-Biehler) $p + iq$ is either $\mathcal{H}_+$-stable or $\mathcal{H}_-$-stable.

Fact: This extends to multivariate stable and real stable polynomials.

We write $p \ll q$ if $p(\boldsymbol{a} \cdot t + \boldsymbol{b}) \ll q(\boldsymbol{a} \cdot t + \boldsymbol{b})$ for all $\boldsymbol{a} \in \mathbb{R}^n_+$ and $\boldsymbol{b} \in \mathbb{R}^n$.
Some interlacing property of the real varieties.

Idea: T preserves real stability $\implies$ $T[ap + bq] = aT[p] = bT[q]$ real stable for all $a, b \in \mathbb{R}$ $\implies$ $T[p] + iT[q] = T[p + iq]$ is $\mathcal{H}_+$- or $\mathcal{H}_-$-stable.

Now: Use the previous slide, plus HKO $\implies$ two-dimensional condition.

Outline

Apolarity theorem for $\mathrm{SU}_n(\mathbb{C})$ form

Grace's theorem: Non-vanishing for $\mathrm{SL}_2(\mathbb{C})$-invariant bilinear form. Another way to prove the BB characterization is by first proving Grace's theorem for $\mathrm{SL}_2(\mathbb{C})^n$. (Recall the product of binomial coeff. in Symb.)

Can we extend this beyond $\mathrm{SL}_2(\mathbb{C})$? There isn't quite an $\mathrm{SL}_n(\mathbb{C})$-invariant form, but there is an $\mathrm{SU}_n(\mathbb{C})$-invariant form for polynomials $p \in \mathbb{C}_h^d[x_1 : \cdots : x_n]$ with $p(x) = \sum_{\boldsymbol{\mu}} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$:

$$\langle p, q \rangle^d := \sum_{|\boldsymbol{\mu}|=d} \binom{d}{\boldsymbol{\mu}}^{-1} p_{\boldsymbol{\mu}} q_{\boldsymbol{\mu}}.$$

Open question: For what classes of polynomials do we get a Grace-type theorem for this bilinear form? (**Also:** Gurvits' capacity conjecture.)

Alternative form: $\langle p, q \rangle^d = D^d(p(\boldsymbol{x})q(\boldsymbol{z}))$ for $D := \sum_{i=1}^n \partial_{x_i} \partial_{z_i}$.