Multivariate Polynomials Exercises

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This set of exercises gives two proofs of the Borcea-Brändén characterization: an alternate proof which does not use polarization, and the proof given in the slides except in the "usual" (non-homogeneous) polynomial spaces. Both proofs give an explication of the conceptual link between a bilinear form and the corresponding symbol of an operator. Further, the proofs are specially written to handle the case of stable polynomials. In particular, the bilinear forms we use below are not quite the bilinear form which shows up in Grace's theorem. However, exercise (1) gives a statement which implies Grace's theorem as a corollary, by considering a "twist" of the bilinear form presented in exercise (2). (See also exercise (7).)

Definition. Given $\boldsymbol{\lambda} \in \mathbb{Z}_+^n$, let $\mathbb{C}_h^{\boldsymbol{\lambda}}[\boldsymbol{x} : \boldsymbol{y}] := \mathbb{C}_h^{(\lambda_1,\ldots,\lambda_n)}[(x_1 : y_1), \ldots, (x_n : y_n)]$ denote the set of polynomials which are homogeneous of degree λ_i in the variables x_i, y_i for all $i \in [n]$. The zeros of polynomials in this space are considered to be in $(\mathbb{CP}^1)^n$. Further, let $\mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}] := \mathbb{C}^{(\lambda_1,\ldots,\lambda_n)}[x_1, \ldots, x_n]$ denote the set of polynomials of degree at most λ_i in the variable x_i for all $i \in [n]$. The zeros of polynomials in this space are considered to be in $\mathbb{C}^n$. Note that these two spaces are isomorphic via per-variable homogenization.

Definition. We denote the complex **upper half-plane** by $\mathcal{H}_+ := \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) > 0\}$. We also let $\mathcal{H}_+$ denote the corresponding subset of $\mathbb{CP}^1$, given by $\mathcal{H}_+ := \{(x : y) \in \mathbb{CP}^1 \ : \ \mathrm{Im}(\frac{x}{y}) > 0\}$. We denote the complex **lower half-plane** by $\mathcal{H}_-$, and define this as a subset of both $\mathbb{C}$ and $\mathbb{CP}^1$ in the analogous way. (We define $\mathrm{Im}(\infty) := 0$, as we consider ∞ to be on the extended real line.)

Definition. Given a polynomial $p \in \mathbb{C}_h^d[x : y]$, the **roots** of p are defined to be the points $(r : s) \in \mathbb{CP}^1$ for which $p(r : s) = 0$. The polynomial p is said to be **stable** if $p(x : y) \neq 0$ whenever $(x : y) \in \mathcal{H}_+$. For $p \in \mathbb{C}_h^{\boldsymbol{\lambda}}[\boldsymbol{x} : \boldsymbol{y}]$, we say that p is **stable** if $p((x_1 : y_1), \ldots, (x_n : y_n)) \neq 0$ whenever $(x_i : y_i) \in \mathcal{H}_+$ for all i. Note that this corresponds to the notion of stable polynomials for $p \in \mathbb{C}^d[x]$ and $p \in \mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$.

Definition. Given $p \in \mathbb{C}^d[x]$, the **polarization** of p is the unique

$$P = \mathrm{Pol}^d(p) \in \mathbb{C}_h^{(1,\ldots,1)}[x_1, \ldots, x_d] = \mathbb{C}^{\mathbf{1}}[\boldsymbol{x}]$$

such that

1. Symmetry: $P(x_1, \ldots, x_d) = P(x_{\sigma(1)}, \ldots, x_{\sigma(d)})$ for all $\sigma \in S_d$.
2. Diagonalization: $p(x) = P(x, x, \ldots, x)$.

For $p \in \mathbb{C}_h^d[x : y]$, the definition is similar for $P = \mathrm{Pol}^d(p) \in \mathbb{C}^{(1,\ldots,1)}[(x_1 : y_1), \ldots, (x_d : y_d)]$. Given $p \in \mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$, the **polarization** of p is defined via

$$\mathrm{Pol}^{\boldsymbol{\lambda}}(p) := \left(\mathrm{Pol}_{x_1}^{\lambda_1} \circ \cdots \circ \mathrm{Pol}_{x_n}^{\lambda_n}\right) p,$$

where $\mathrm{Pol}_{x_i}^{\lambda_i}$ indicates applying the polarization operator defined above to the variable x_i This definition then extends to $p \in \mathbb{C}_h^{\boldsymbol{\lambda}}[\boldsymbol{x} : \boldsymbol{y}]$ in the natural way.

Example. Let $\boldsymbol{\lambda} = (2,2)$ and define $p(x,z) := 4x^2z + 4xz + 4$. Then the polarization is:

$$\begin{aligned} P(x_1, x_2, z_1, z_2) &= 2x_1x_2z_1 + 2x_1x_2z_2 \\ &\quad + x_1z_1 + x_1z_2 + x_2z_1 + x_2z_2 \\ &\quad + 4. \end{aligned}$$

For the equivalent homogeneous polynomial $p(x,z) := 4x^2zw + 4xyzw + 4y^2w^2$, the polarization is similar:

$$\begin{aligned} P((x_1 : y_1), (x_2 : y_2), (z_1 : w_1), (z_2 : w_2)) &= 2x_1x_2z_1w_2 + 2x_1x_2z_2w_1 \\ &\quad + x_1y_2z_1w_2 + x_1y_2z_2w_1 + x_2y_1z_1w_2 + x_2y_1z_2w_1 \\ &\quad + 4y_1y_2w_1w_2. \end{aligned}$$

Exercises

A proof of the Borcea-Brändén characterization without polarization

1. Given $p \in \mathbb{C}_h^{(d,d)}[t : s, z : w]$, prove that $(\partial_t\partial_w + \partial_s\partial_z)p$ is stable (or identically 0) if p is stable. Here is one possible outline of the proof.

   (a) Given a stable polynomial $p \in \mathbb{C}_h^d[t : s]$, define a map

   $$F_p : \mathcal{H}_+ \to \mathbb{CP}^1 \quad \text{given by} \quad (t : s) \mapsto (\partial_s p(t : s) : \partial_t p(t : s)).$$

   Prove that either F_p maps the upper half-plane $\mathcal{H}_+$ into itself, or there exists $a, b \in \mathbb{R}$, not both 0, such that $(a\partial_t + b\partial_s)p \equiv 0$. (**Hint:** There is a way to frame this as an equivalent form of Laguerre's theorem for stable polynomials.)

   (b) Given a stable polynomial $p \in \mathbb{C}_h^{(d,d)}[t : s, z : w]$, define a map

   $$G_p : \mathcal{H}_+^2 \to \mathbb{CP}^1 \quad \text{given by} \quad (t : s, z : w) \mapsto ((\partial_t\partial_w + \partial_s\partial_z)p(t : s, z : w) : \partial_t\partial_z p(t : s, z : w)).$$

   Prove that either G_p maps $\mathcal{H}_+^2$ into $\mathcal{H}_+$, or there exists $a, b \in \mathbb{R}$, not both 0, such that either $(a\partial_t + b\partial_s)\partial_z p \equiv 0$ or $(a\partial_z + b\partial_w)\partial_t p \equiv 0$.

   (c) Handle the details to complete the proof. (**Hint:** If $\partial_t\partial_z p \equiv 0$, perhaps $\partial_s\partial_w p \not\equiv 0$?)

2. Define a bilinear form

   $$B(p,q) := \frac{1}{d!^2}(\partial_t\partial_w + \partial_s\partial_z)^d \left[p(t : s) \cdot q(z : w)\right]$$

   for $p, q \in \mathbb{C}_h^d[x : y]$. Let $\mathcal{L}$ denote the space of linear endomorphisms of the vector space $\mathbb{C}_h^d[x : y]$. Consider the linear isomorphism:

   $$\mathcal{L} \cong \mathbb{C}_h^d[x : y] \otimes \mathbb{C}_h^d[x : y]^* \cong \mathbb{C}_h^d[x : y] \otimes \mathbb{C}_h^d[x : y] \cong \mathbb{C}_h^{(d,d)}[x : y, z : w],$$

   where the first isomorphism is the canonical one, the second isomorphism is given on simple tensors by $p \otimes B(q, \cdot) \mapsto p \otimes q$, and the third isomorphism is given by $x^jy^{d-j} \otimes x^ky^{d-k} \mapsto x^jy^{d-j}z^kw^{d-k}$. Let this chain of isomorphisms be denoted by $\Phi : \mathcal{L} \to \mathbb{C}_h^{(d,d)}[x : y, z : w]$. (Here, $\Phi[T]$ is the **symbol of the operator** T, and is usually denoted $\mathrm{Symb}[T]$.)

   Prove the formula

   $$\Phi[T](x : y, z : w) = \sum_{\ell=0}^{d} \binom{d}{\ell} z^{d-\ell}w^\ell \cdot T\left[x^\ell y^{d-\ell}\right] = T\left[(xw + yz)^d\right],$$

   where in the last expression, T only acts on the $x : y$ variables. One option is to try to prove this by directly computing $\Phi[T]$, which is not too hard. Another option is to prove it on a basis of $\mathcal{L}$ as follows:

(a) Prove that the linear operator $T := \Phi^{-1}(x^j y^{d-j} z^k w^{d-k})$ is given by

$$T\left[\binom{d}{\ell} x^\ell y^{d-\ell}\right] = \begin{cases} x^j y^{d-j} & \ell = d-k \\ 0 & \ell \neq d-k \end{cases}.$$

(b) Show how this implies the above expression for $\Phi[T]$ in full generality.

3. Using the notation of the previous exercise, prove that for any fixed $T \in \mathcal{L}$ and any fixed $(x:y) \in \mathbb{CP}^1$, we have that

$$T[p](x:y) = B\big(\Phi[T](x:y, z:w), p(z:w)\big),$$

where the bilinear form B acts on the polynomials in terms of the variables $(z:w)$.

4. **(The Borcea-Brändén characterization for univariate stable polynomials)** Use exercises (1), (2), and (3) to prove the following: If T is a linear endomorphism on $\mathbb{C}_h^d[x:y]$ such that $T\left[(xw+yz)^d\right]$ is a stable polynomial, then $T[p]$ is either stable or identically 0 whenever p is stable.

   Show that this implies the equivalent statement for $\mathbb{C}^d[x]$: If T is a linear endomorphism on $\mathbb{C}^d[x]$ such that $T\left[(x+z)^d\right]$ is a stable polynomial, then $T[p]$ is either stable or identically 0 whenever p is stable.

5. **(The Borcea-Brändén characterization for real-rooted polynomials)** Using the previous exercise, prove the following: If T is a linear endomorphism on $\mathbb{R}^d[x]$ such that either $T\left[(x+z)^d\right]$ or $T\left[(xz+1)^d\right]$ is a stable polynomial, then $T[p]$ is either real-rooted or identically 0 whenever p is real-rooted.

6. **(The Borcea-Brändén characterization for multivariate stable and real stable polynomials)** Define a bilinear form

$$B(p,q) := \frac{1}{\lambda_1!^2 \cdots \lambda_n!^2} \prod_{i=1}^n (\partial_{t_i}\partial_{w_i} + \partial_{s_i}\partial_{z_i})^{\lambda_i}\left[p(\boldsymbol{t}:\boldsymbol{s}) \cdot q(\boldsymbol{z}:\boldsymbol{w})\right]$$

   for $p, q \in \mathbb{C}_h^{\boldsymbol{\lambda}}[\boldsymbol{x}:\boldsymbol{y}]$. Show that by defining Φ in the same way as above, we have for any linear endomorphism T on $\mathbb{C}_h^{\boldsymbol{\lambda}}[\boldsymbol{x}:\boldsymbol{y}]$ that

$$\Phi[T](\boldsymbol{x}:\boldsymbol{y}, \boldsymbol{z}:\boldsymbol{w}) = \sum_{\boldsymbol{0} \leq \boldsymbol{\mu} \leq \boldsymbol{\lambda}} \binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}} \boldsymbol{z}^{\boldsymbol{\lambda}-\boldsymbol{\mu}} \boldsymbol{w}^{\boldsymbol{\mu}} \cdot T\left[\boldsymbol{x}^{\boldsymbol{\mu}} \boldsymbol{y}^{\boldsymbol{\lambda}-\boldsymbol{\mu}}\right] = T\left[\prod_{i=1}^n (x_i w_i + y_i z_i)^{\lambda_i}\right],$$

   where $\binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}} = \prod_i \binom{\lambda_i}{\mu_i}$ and $\boldsymbol{x}^{\boldsymbol{\mu}} = \prod_i x_i^{\mu_i}$, and in the last expression T only acts on the $\boldsymbol{x}:\boldsymbol{y}$ variables. Further show that we have the formula

$$T[p](\boldsymbol{x}:\boldsymbol{y}) = B\big(\Phi[T](\boldsymbol{x}:\boldsymbol{y}, \boldsymbol{z}:\boldsymbol{w}), p(\boldsymbol{z}:\boldsymbol{w})\big),$$

   where the bilinear form B acts on the polynomials in terms of the variables $(\boldsymbol{z}:\boldsymbol{w})$. State and prove the results in this case which are analogous to that of exercises (4) and (5).

   Aside: When T is a linear endomorphism on $\mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$, then $\Phi[T]$ should be replaced by $T\left[\prod_{i=1}^n (x_i+z_i)^{\lambda_i}\right]$ and/or $T\left[\prod_{i=1}^n (x_i z_i + 1)^{\lambda_i}\right]$ in the real stable case.

A proof of the Borcea-Brändén characterization with "usual" (non-homogeneous) polynomials

7. Define a bilinear form

$$B(p,q) := \prod_{i=1}^n (\partial_{t_i} + \partial_{z_i})\left[p(\boldsymbol{t}) \cdot q(\boldsymbol{z})\right]$$

for $p, q \in \mathbb{C}^{\mathbf{1}}[\boldsymbol{x}]$. Let $\mathcal{L}$ denote the space of linear endomorphisms of the vector space $\mathbb{C}^{\mathbf{1}}[\boldsymbol{x}]$. Consider the linear isomorphism:

$$\mathcal{L} \cong \mathbb{C}^{\mathbf{1}}[\boldsymbol{x}] \otimes \mathbb{C}^{\mathbf{1}}[\boldsymbol{x}]^* \cong \mathbb{C}^{\mathbf{1}}[\boldsymbol{x}] \otimes \mathbb{C}^{\mathbf{1}}[\boldsymbol{x}] \cong \mathbb{C}^{(1,\ldots,1,1,\ldots,1)}[\boldsymbol{x}, \boldsymbol{z}],$$

where the first isomorphism is the canonical one, the second isomorphism is given on simple tensors by $p \otimes B(q, \cdot) \mapsto p \otimes q$, and the third isomorphism is given by $x^j \otimes x^k \mapsto x^j z^k$. Let this chain of isomorphisms be denoted by $\Phi : \mathcal{L} \to \mathbb{C}^{(\mathbf{1},\mathbf{1})}[\boldsymbol{x}, \boldsymbol{z}]$. (Here, $\Phi[T]$ is the **symbol of the operator** T, and is usually denoted $\mathrm{Symb}[T]$.)

Prove that for any T, we have

$$\Phi[T](\boldsymbol{x}, \boldsymbol{z}) = \sum_{\mathbf{0} \leq \boldsymbol{\mu} \leq \mathbf{1}} \boldsymbol{z}^{\mathbf{1}-\boldsymbol{\mu}} \cdot T\left[\boldsymbol{x}^{\boldsymbol{\mu}}\right] = T\left[\prod_{i=1}^{n} (x_i + z_i)\right],$$

where $\boldsymbol{x}^{\boldsymbol{\mu}} = \prod_i x_i^{\mu_i}$ and in the last expression T only acts on the $\boldsymbol{x}$ variables. (**Hint:** See exercise 2.)

8. Using the notation of the previous exercise, prove that for any fixed $T \in \mathcal{L}$ and any fixed $\boldsymbol{x} \in \mathbb{C}^n$, we have that

$$T[p](\boldsymbol{x}) = B\big(\Phi[T](\boldsymbol{x}, \boldsymbol{z}), p(\boldsymbol{z})\big),$$

where the bilinear form B acts on the polynomials in terms of the variables $\boldsymbol{z}$.

9. **(The Borcea-Brändén characterization for multiaffine stable polynomials)** Use exercises (8) and (9) to prove the following: If T is a linear endomorphism on $\mathbb{C}^{\mathbf{1}}[\boldsymbol{x}]$ such that $T\left[\prod_{i=1}^n (x_i + z_i)\right]$ is a stable polynomial, then $T[p]$ is either stable or identically 0 whenever p is stable.

10. **(The Borcea-Brändén characterization for multivariate stable polynomials)** Recall that one corollary of the Walsh coincidence theorem is that $\mathrm{Pol}^d(p)$ is stable if and only if p is stable for $p \in \mathbb{C}^d[x]$. Use this fact and the previous exercise to prove the following: If T is a linear endomorphism on $\mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ such that $T\left[\prod_{i=1}^n (x_i + z_i)^{\lambda_i}\right]$ is a stable polynomial, then $T[p]$ is either stable or identically 0 whenever p is stable.

11. **(The Borcea-Brändén characterization for multivariate real stable polynomials)** Using the previous exercise, prove the following: If T is a linear endomorphism on $\mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ such that either $T\left[\prod_{i=1}^n (x_i + z_i)^{\lambda_i}\right]$ or $T\left[\prod_{i=1}^n (x_i z_i + 1)^{\lambda_i}\right]$ is a stable polynomial, then $T[p]$ is either real stable or identically 0 whenever p is real stable.

Other exercises

12. **(Multivariate Grace's theorem)** Prove the multivariate Grace's theorem: Given $p, q \in \mathbb{C}_h^{\boldsymbol{\lambda}}[\boldsymbol{x} : \boldsymbol{y}]$, if there is a circular region $C \subset \mathbb{CP}^1$ for which p is C^n-stable and q is $(\mathbb{CP}^1 \setminus C)^n$-stable, then

$$\prod_{i=1}^{n} (\partial_{t_i} \partial_{w_i} - \partial_{s_i} \partial_{z_i})^{\lambda_i} \left[p(\boldsymbol{t} : \boldsymbol{s}) \cdot q(\boldsymbol{z} : \boldsymbol{w})\right] \neq 0.$$

(**Hint:** What invariance properties does this bilinear form have?)

13. Prove that the linear operator $1 - \partial_{x_1} \partial_{x_2}$ acting on $\mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ preserves real stability for any $\boldsymbol{\lambda}$.

14. Prove that the linear projection $\mathrm{MAP} : \mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}] \to \mathbb{R}^{\mathbf{1}}[\boldsymbol{x}]$, which acts as the identity on multiaffine terms and annihilates all other terms, preserves real stability for any $\boldsymbol{\lambda}$.

15. Prove that a polynomial $p \in \mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ is stable if and only if the real part p_R of p and the imaginary part p_I of p are both real stable and $p_I \ll p_R$.

16. A polynomial in $\mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ is **strictly stable** if it is C^n-stable where $C = \overline{\mathcal{H}_+}$ is the closure of the upper half-plane. Prove that the set of stable polynomials in $\mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ is the closure of the (open) set of all strictly stable polynomials in $\mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$.

17. Prove that any linear subspace of $\mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ consisting of stable polynomials is at most one-dimensional. Prove that any linear subspace of $\mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ consisting of real stable polynomials is at most two-dimensional. (**Hint:** Do the real stable one first.)

18. Prove the other direction of the Borcea-Brändén characterization:

    (a) If T is a linear operator on $\mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ which preserves stability and such that the image of T is of dimension greater than 1, then the symbol of T (that is, $\Phi(T)$ in our notation here) is a stable polynomial.

    (b) If T is a linear operator on $\mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ which preserves real stability and such that the image of T is of dimension greater than 2, then the symbol of T (that is, $\Phi(T)$ in our notation here) is a real stable polynomial.

19. Everything we have done here has been in terms of spaces of polynomials which have bounded degree. There is also a Borcea-Brändén characterization in terms of linear operators on the whole space of polynomials in a certain number of variables (that is, on $\mathbb{C}[\boldsymbol{x}]$ or $\mathbb{R}[\boldsymbol{x}]$). What sort of symbol $\Phi[T]$ would be needed for this kind of characterization? For those who haven't already seen it: any guesses on what this symbol is?

20. Use any of the above versions of the Borcea-Brändén characterization to give a similar result for C^n-stability where C is the open complex unit disc.