Completely Log-concave (Lorentzian) Polynomials

Polynomial Capacity: Theory, Applications, Generalizations

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Notation

Polynomial notation:

- $\mathbb{R}, \mathbb{R}_+, \mathbb{C}, \mathbb{Z}_+ :=$ reals, non-negative reals, complex numbers, non-negative integers.
- $\boldsymbol{x}^{\boldsymbol{\mu}} := \prod_i x_i^{\mu_i}$ and $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ is entrywise.
- $\mathbb{R}[\boldsymbol{x}] :=$ v.s. of real polynomials in n variables.
- $\mathbb{R}_+[\boldsymbol{x}] :=$ v.s. of real polynomials with non-negative coefficients.
- $\mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}] :=$ v.s. of polynomials of degree at most λ_i in x_i.
- For $p \in \mathbb{R}[\boldsymbol{x}]$, we write $p(\boldsymbol{x}) = \sum_{\boldsymbol{\mu}} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$.
- For d-homogeneous $p \in \mathbb{R}[\boldsymbol{x}]$, we write $p(\boldsymbol{x}) = \sum_{|\boldsymbol{\mu}|=d} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$.
- The **support** of p is the set of $\boldsymbol{\mu} \in \mathbb{Z}_+^n$ for which $p_{\boldsymbol{\mu}} \neq 0$.
- $\frac{d}{dx} = \frac{\partial}{\partial x} = \partial_x :=$ derivative with respect to x, and $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} := \prod_i \partial_{x_i}^{\mu_i}$.
- $p(\boldsymbol{a} \cdot t + \boldsymbol{b}) = p(a_1 t + b_1, \ldots, a_n t + b_n) \in \mathbb{R}^{\lambda_1 + \cdots + \lambda_n}[t]$ is a **linear restriction** of the polynomial $p \in \mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$, where $\boldsymbol{a} \in \mathbb{R}_+^n$ and $\boldsymbol{b} \in \mathbb{R}^n$.

Outline

1. Completely log-concave (Lorentzian) polynomials
   - Motivation
   - Definition
2. Properties of completely log-concave polynomials
   - Log-concavity and Lorentz signature
   - Products of CLC polynomials
3. Reduction to quadratics
   - Proof of the reduction
   - Corollaries of the reduction
4. Symbol theorem for CLC polynomials

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Motivation: Matroid basis-generating polynomials

The spanning trees of G = **set of bases of a graphic matroid**.

Matroid: $M = (E, \mathcal{I})$ where E is the **ground set** and $\mathcal{I} \subseteq 2^E$ are the **independent** subsets, which satisfy:

1. **Nonempty:** $\mathcal{I} \neq \varnothing$.
2. **Hereditary:** $B \in \mathcal{I}$ and $A \subseteq B$ implies $A \in \mathcal{I}$.
3. **Exchange/Augmentation:** For all $A, B \in \mathcal{I}$ such that $|A| < |B|$, there exists $e \in B \setminus A$ such that $A \cup \{e\} \in \mathcal{I}$.

E.g.: A set of vectors in a vector space, with $\mathcal{I}$ given by linearly independent subsets (**linear matroid**). The set of edges of a graph, with $\mathcal{I}$ given by subsets with no cycles (**graphic matroid**). **Many more...**

Maximal $B \in \mathcal{I}$ are the **bases**, $\mathcal{B} \subset \mathcal{I}$, of M. **Another definition of M:**

3. **Exchange:** For any bases $B_1, B_2 \in \mathcal{B}$ and any $e_1 \in B_1 \setminus B_2$, there exists $e_2 \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}$.

The spanning tree polynomial is a basis-generating polynomial. **Others?**

Motivation: Hodge-Riemann relations

Adiprasito-Huh-Katz '15: Resolution of the Heron-Rota-Welsh conjecture saying that the coefficients of the characteristic polynomial of a matroid form a log-concave sequence.

Use Hodge-Riemann (HR) relations: These are certain definiteness properties related to various linear maps and their kernels.

Appear in many contexts:

- Cohomology of real forms on compact Kähler manifold [Gromov '90].
- Algebraic cycles modulo homological equivalence on a smooth projective variety [Grothendieck '69].
- McMullen's algebra generated by a simple convex polytope ['93].

The part that is used for the Heron-Rota-Welsh conjecture boils down to a certain quadratic form having **Lorentz signature** $(+, -, -, \ldots, -)$.

Fact: Hessians of a real stable polynomial $p \in \mathbb{R}_+[\boldsymbol{x}]$ in the positive orthant all have Lorentz signature. **Char. polynomial is not real-rooted.**

Motivation: Mason's strongest conjecture

Conjecture [Mason '75]: If $M = (E, \mathcal{I})$ is a matroid such that $|E| = n$, and I_k denotes the number of independent sets of M of size k, then $(I_k)_{k=0}^n$ forms an ultra log-concave sequence.

Easy idea: Let's use Newton's inequalities. Need to show that

$$I_M(t) := \sum_{k=0}^{n} I_k t^k$$

is a real-rooted polynomial. $\implies$ ULC: $\frac{I_{k+1}}{\binom{n}{k+1}} \cdot \frac{I_{k-1}}{\binom{n}{k-1}} \leq \left[\frac{I_k}{\binom{n}{k}}\right]^2$.

Problem: Size of maximum independent set **can be less than** n.
$\implies \deg(I_M(t)) =: d < n$. $\implies$ ULC definition **changes**.

Fact: There is an M such that $\deg(I_M(t)) = d < n$, but the coefficients are **not** ULC with respect to degree d. $\implies$ $I_M(t)$ is **not real-rooted**.

Motivation: Random walks on simplicial complexes

Simplicial complex: Collection X of subsets of E for which $\sigma \in X$ and $\tau \subset \sigma$ implies $\tau \in X$. **E.g.:** Matroids are simplicial complexes.

Local random walks: Random walk on the "1-skeleton" of a given $\sigma \in X$.

Kaufman-Oppenheim '18: Random walk on simplicial complex has large spectral gap if the second largest eigenvalue of the random walk matrix of every 1-skeleton is small (largest eigenvalue is 1).

The point: Large spectral gap implies rapid mixing of the random walk, which implies efficient sampling/counting.

How is this related to polynomials? The 1-skeletons can be associated to multiaffine polynomials, where the choice of σ corresponds to a choice of derivatives. The random walk matrix is related to the Hessian matrix.

Real stable polynomials: Hessians have Lorentz signature.
$\implies$ Second largest Hessian eigenvalue ≤ 0. **Other polynomials?**

Completely log-concave polynomials

Idea: Combine Lorentz signature with ULC coefficient conditions.
Also: Want partial derivatives to preserve the property.

Definition (Gurvits '09, Anari-Oveis Gharan-Vinzant '19)

A d-homogeneous polynomial $p \in \mathbb{R}_+[\boldsymbol{x}]$ is **completely log-concave (CLC)** if for any choice of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}_+^n$ for any k, we have that

$$\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} p := \left(\sum_i v_{1i} \partial_{x_i} \right) \cdots \left(\sum_i v_{ki} \partial_{x_i} \right) p$$

is log-concave in the positive orthant or $\equiv 0$.

Fact: If $d = 1$, linear with non-negative coefficients $\implies$ trivial.
Fact: If $n = 2$, CLC is equivalent to ultra log-concave coefficients.
Fact: If $d = 2$, CLC is equivalent to real stability.
E.g.: Matroid basis polynomials, $\mathrm{vol}(\sum_i x_i K_i)$ for convex compact K_i

Equivalent theory of **Lorentzian** polynomials [Brändén-Huh '19] involves matroidal support. (Many equivalent definitions.)

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Properties of completely log-concave (CLC) polynomials

Proposition

If $p, q \in \mathbb{R}_+[\boldsymbol{x}]$ are CLC polynomials, then

1. $\nabla_{\boldsymbol{a}} p$ for $\boldsymbol{a} \in \mathbb{R}_+^n$ and $p|_{x_i=0}$ are CLC.
2. The Hessian $\nabla^2 p(\boldsymbol{a})$ is Lorentz for all $\boldsymbol{a} \in \mathbb{R}_+^n$.
3. $p(A\boldsymbol{x})$ is CLC for all $n \times m$ matrices A with non-negative entries.
4. $p(\boldsymbol{a} \cdot t + \boldsymbol{b} \cdot s) \in \mathbb{R}_+[t, s]$ is CLC for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}_+^n$.
5. $p(\boldsymbol{x}) \cdot q(\boldsymbol{z}) \in \mathbb{R}_+[\boldsymbol{x}, \boldsymbol{z}]$ is CLC.
6. $p(\boldsymbol{x}) \cdot q(\boldsymbol{x}) \in \mathbb{R}_+[\boldsymbol{x}]$ is CLC.

Lorentz matrix: Hermitian with signature $(+, -, -, \ldots, -)$, or in closure.

Note: (1) is straightforward, and (4) follows from (3).
Also: (6) follows from (3) and (5), via $f(\boldsymbol{x}, \boldsymbol{z}) := p(\boldsymbol{x}) \cdot q(\boldsymbol{z})$ and

$$p(\boldsymbol{x}) \cdot q(\boldsymbol{x}) = f(A\boldsymbol{x}) := f\left(\begin{bmatrix} I_n & I_n \end{bmatrix}^\top \boldsymbol{x}\right).$$

Log-concavity and Lorentz signature

Lorentz equivalence: Let $p \in \mathbb{R}_+[\boldsymbol{x}]$ be d-homogeneous, and fix some $\boldsymbol{a} \in \mathbb{R}_+^n$ with $p(\boldsymbol{a}) > 0$. Let $Q := \nabla^2 p(\boldsymbol{a})$ denote the Hessian at $\boldsymbol{a}$. TFAE:

1. $\nabla^2 \log p(\boldsymbol{a})$ is negative semidefinite (log-concavity at $\boldsymbol{a}$).
2. Q is negative semidefinite on $(Q\boldsymbol{a})^\perp$.
3. Q is negative semidefinite on $(Q\boldsymbol{b})^\perp$ for all $\boldsymbol{b} \in \mathbb{R}_+^n$ with $p(\boldsymbol{b}) > 0$.
4. Q is negative semidefinite on some $(n-1)$-dimensional subspace.
5. Q is Lorentz.

Euler's identity: $d \cdot p = \sum_i x_i \partial_{x_i} p$. Apply to $\partial_{x_j} p$ and p to get

$$Q\boldsymbol{a} = (d-1) \cdot \nabla p(\boldsymbol{a}) \qquad \text{and} \qquad \boldsymbol{a}^\top Q \boldsymbol{a} = d(d-1) \cdot p(\boldsymbol{a}).$$

E.g.: $(Q\boldsymbol{a})_j = \sum_i a_i \partial_{x_i} \partial_{x_j} p(\boldsymbol{a}) = (d-1) \cdot \partial_{x_j} p(\boldsymbol{a})$.

This implies $\nabla^2 \log p(\boldsymbol{a})$ can be written as

$$\left. \frac{p \cdot \nabla^2 p - (\nabla p) \cdot (\nabla p)^\top}{p^2} \right|_{\boldsymbol{x}=\boldsymbol{a}} = d(d-1) \cdot \frac{(\boldsymbol{a}^\top Q \boldsymbol{a}) Q - \frac{d}{d-1} (Q\boldsymbol{a})(Q\boldsymbol{a})^\top}{(\boldsymbol{a}^\top Q \boldsymbol{a})^2}.$$

Log-concavity and Lorentz signature

To prove: For $Q := \nabla^2 p(\boldsymbol{a})$, the following are equivalent:

1. $\nabla^2 \log p(\boldsymbol{a})$ is negative semidefinite (log-concavity at $\boldsymbol{a}$).
2. Q is negative semidefinite on $(Q\boldsymbol{a})^\perp$.
3. Q is negative semidefinite on $(Q\boldsymbol{b})^\perp$ for all $\boldsymbol{b} \in \mathbb{R}^n_+$ with $p(\boldsymbol{b}) > 0$.
4. Q is negative semidefinite on some $(n-1)$-dimensional subspace.
5. Q is Lorentz.

Recall: $\nabla^2 \log p(\boldsymbol{a}) \cong (\boldsymbol{a}^\top Q\boldsymbol{a}) \cdot Q - \frac{d}{d-1} \cdot (Q\boldsymbol{a})(Q\boldsymbol{a})^\top$ and $\boldsymbol{a}^\top Q\boldsymbol{a} > 0$.

(1) $\implies$ (2): First, for all $\boldsymbol{z} \in (Q\boldsymbol{a})^\perp \iff \boldsymbol{z}^\top Q\boldsymbol{a} = 0$ we have

$$0 \geq \boldsymbol{z}^\top \left[(\boldsymbol{a}^\top Q\boldsymbol{a}) \cdot Q - \frac{d}{d-1} \cdot (Q\boldsymbol{a})(Q\boldsymbol{a})^\top\right] \boldsymbol{z} = (\boldsymbol{a}^\top Q\boldsymbol{a}) \cdot (\boldsymbol{z}^\top Q\boldsymbol{z}).$$

(2) $\implies$ (4): $(Q\boldsymbol{a})^\top$ is an $(n-1)$-dimensional subspace.

(4) $\implies$ (5): By assumption Q has at most one positive eigenvalue. Since entries of Q are non-negative, Q has at least one non-negative eigenvalue.

Log-concavity and Lorentz signature

To prove: For $Q := \nabla^2 p(\boldsymbol{a})$, the following are equivalent:

1. $\nabla^2 \log p(\boldsymbol{a})$ is negative semidefinite (log-concavity at $\boldsymbol{a}$).
2. Q is negative semidefinite on $(Q\boldsymbol{a})^\perp$.
3. Q is negative semidefinite on $(Q\boldsymbol{b})^\perp$ for all $\boldsymbol{b} \in \mathbb{R}^n_+$ with $p(\boldsymbol{b}) > 0$.
4. Q is negative semidefinite on some $(n-1)$-dimensional subspace.
5. Q is Lorentz.

Recall: $\nabla^2 \log p(\boldsymbol{a}) \cong (\boldsymbol{a}^\top Q\boldsymbol{a}) \cdot Q - \frac{d}{d-1} \cdot (Q\boldsymbol{a})(Q\boldsymbol{a})^\top$ and $\boldsymbol{a}^\top Q\boldsymbol{a} > 0$.

(5) $\implies$ (1): Let P be the $n \times 2$ matrix with columns $\boldsymbol{a}$ and $\boldsymbol{z} \in \mathbb{R}^n$:

$$P^\top QP = \begin{bmatrix} \boldsymbol{a}^\top Q\boldsymbol{a} & \boldsymbol{a}^\top Q\boldsymbol{z} \\ \boldsymbol{z}^\top Q\boldsymbol{a} & \boldsymbol{z}^\top Q\boldsymbol{z} \end{bmatrix} \implies \det(P^\top QP) \gtrsim \boldsymbol{z}^\top [\nabla^2 \log p(\boldsymbol{a})]\boldsymbol{z}.$$

So: Want to show $\det(P^\top QP) \leq 0$. **Enough to show** $P^\top QP$ is not PD, since $\boldsymbol{a}^\top Q\boldsymbol{a} > 0$ implies $P^\top QP$ is not NSD.

But: $P^\top QP$ cannot be PD, or else Q would have two positive eigenvalues.

Log-concavity and Lorentz signature

To prove: For $Q := \nabla^2 p(\boldsymbol{a})$, the following are equivalent:

1. $\nabla^2 \log p(\boldsymbol{a})$ is negative semidefinite (log-concavity at $\boldsymbol{a}$).
2. Q is negative semidefinite on $(Q\boldsymbol{a})^\perp$.
3. Q is negative semidefinite on $(Q\boldsymbol{b})^\perp$ for all $\boldsymbol{b} \in \mathbb{R}_+^n$ with $p(\boldsymbol{b}) > 0$.
4. Q is negative semidefinite on some $(n-1)$-dimensional subspace.
5. Q is Lorentz.

Recall: $\nabla^2 \log p(\boldsymbol{a}) \cong (\boldsymbol{a}^\top Q\boldsymbol{a}) \cdot Q - \frac{d}{d-1} \cdot (Q\boldsymbol{a})(Q\boldsymbol{a})^\top$ and $\boldsymbol{a}^\top Q\boldsymbol{a} > 0$.

(2) $\implies$ (3): Note that both conditions only depend on the matrix Q. Consider the polynomial $q(\boldsymbol{x}) := \frac{1}{2}\boldsymbol{x}^\top Q\boldsymbol{x}$, which is such that

$$\nabla^2 q(\boldsymbol{a}) = Q = \nabla^2 q(\boldsymbol{b}) \qquad \text{for all } \boldsymbol{b}.$$

So applying the equivalence (2) $\iff$ (4) to q and $Q' := \nabla^2 q(\boldsymbol{b})$ says that Q' is negative semidefinite on $(Q'\boldsymbol{b})^\perp$ since $Q' = Q$ is negative semidefinite on the $(n-1)$-dimensional subspace $(Q\boldsymbol{a})^\perp$.

Properties of CLC polynomials, revisited

Proposition

If $p, q \in \mathbb{R}_+[\boldsymbol{x}]$ are CLC polynomials, then

1. $\nabla_{\boldsymbol{a}} p$ for $\boldsymbol{a} \in \mathbb{R}_+^n$ and $p|_{x_i=0}$ are CLC.
2. The Hessian $\nabla^2 p(\boldsymbol{a})$ is Lorentz for all $\boldsymbol{a} \in \mathbb{R}_+^n$.
3. $p(A\boldsymbol{x})$ is CLC for all $n \times m$ matrices A with non-negative entries.
4. $p(\boldsymbol{a} \cdot t + \boldsymbol{b} \cdot s) \in \mathbb{R}_+[t, s]$ is CLC for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}_+^n$.
5. $p(\boldsymbol{x}) \cdot q(\boldsymbol{z}) \in \mathbb{R}_+[\boldsymbol{x}, \boldsymbol{z}]$ is CLC.
6. $p(\boldsymbol{x}) \cdot q(\boldsymbol{x}) \in \mathbb{R}_+[\boldsymbol{x}]$ is CLC.

Bonus: A quadratic homogeneous polynomial $p(\boldsymbol{x}) = \boldsymbol{x}^\top A\boldsymbol{x}$ is CLC if and only if A is Lorentz.

Next: $p(A\mathbf{x})$ and products.

Precomposition by positive linear action preserves CLC

Fact: If $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ is CLC and A is an $n \times m$ matrix with non-negative entries, then $p(A\boldsymbol{x}) \in \mathbb{R}_+[x_1, \ldots, x_m]$ is CLC.

Proof: For any $\boldsymbol{v} \in \mathbb{R}_+^m$, we have

$$\nabla_{\boldsymbol{v}} \left[p(A\boldsymbol{x}) \right] = \sum_{j=1}^m v_j \partial_{x_j} \left[p \left(\sum_{k=1}^m a_{1k} x_k, \ldots, \sum_{k=1}^m a_{nk} x_k \right) \right]$$

$$= \sum_{j=1}^m v_j \left[\left(\sum_{i=1}^n a_{ij} \partial_{x_i} \right) p \right] (A\boldsymbol{x}) = \left[\left(\sum_{i=1}^n \sum_{j=1}^m a_{ij} v_j \partial_{x_i} \right) p \right] (A\boldsymbol{x}).$$

To complete the proof, need to show that $p(A\boldsymbol{x})$ is log-concave in the positive orthant whenever p is:

$$\log p(A[t \cdot \boldsymbol{x} + (1-t) \cdot \boldsymbol{y}]) = \log p(t \cdot (A\boldsymbol{x}) + (1-t) \cdot (A\boldsymbol{y}))$$

$$\geq t \cdot \log p(A\boldsymbol{x}) + (1-t) \cdot \log p(A\boldsymbol{y}).$$

Products of CLC polynomials are CLC

Lemma (sum-of-CLCs): If $p, q \in \mathbb{R}_+[\boldsymbol{x}]$ are d-homog. CLC polynomials such that $\nabla_{\boldsymbol{b}} p = \nabla_{\boldsymbol{c}} q \not\equiv 0$ for some $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}_+^n$, then $p + q$ is CLC.

Corollary: If $p(\boldsymbol{x})$ and $q(\boldsymbol{z})$ are CLC, then so is $p(\boldsymbol{x}) \cdot q(\boldsymbol{z})$.

Proof: Log-concavity is straightforward, since the log of a product is the sum of logs. By induction, for any $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}_+^n$

$$\nabla_{(\boldsymbol{b},\boldsymbol{c})} [p(\boldsymbol{x}) \cdot q(\boldsymbol{z})] = \nabla_{\boldsymbol{b}} p(\boldsymbol{x}) \cdot q(\boldsymbol{z}) + p(\boldsymbol{x}) \cdot \nabla_{\boldsymbol{c}} q(\boldsymbol{z})$$

is a sum of CLC polynomials. Further,

$$\nabla_{(\boldsymbol{0},\boldsymbol{c})} [\nabla_{\boldsymbol{b}} p(\boldsymbol{x}) \cdot q(\boldsymbol{z})] = \nabla_{\boldsymbol{b}} p(\boldsymbol{x}) \cdot \nabla_{\boldsymbol{c}} q(\boldsymbol{z}) = \nabla_{(\boldsymbol{b},\boldsymbol{0})} [p(\boldsymbol{x}) \cdot \nabla_{\boldsymbol{c}} q(\boldsymbol{z})] .$$

Therefore the sum-of-CLCs lemma applies if $\nabla_{\boldsymbol{b}} p(\boldsymbol{x}) \cdot \nabla_{\boldsymbol{c}} q(\boldsymbol{z}) \not\equiv 0$. If $\nabla_{\boldsymbol{b}} p(\boldsymbol{x}) \cdot \nabla_{\boldsymbol{c}} q(\boldsymbol{z}) \equiv 0$, then one of the polynomials in the above sum is 0.

Proof of the sum-of-CLCs lemma

Lemma (sum-of-CLCs): If $p, q \in \mathbb{R}_+[\boldsymbol{x}]$ are d-homog. CLC polynomials such that $\nabla_{\boldsymbol{b}} p = \nabla_{\boldsymbol{c}} q \not\equiv 0$ for some $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^n_+$, then $p + q$ is CLC.

Proof of Lemma: By induction on degree, for all $\boldsymbol{a} \in \mathbb{R}^n_{>0}$ we have

$$\nabla_{\boldsymbol{b}}(\nabla_{\boldsymbol{a}} p) = \nabla_{\boldsymbol{c}}(\nabla_{\boldsymbol{a}} q) \quad \Longrightarrow \quad \nabla_{\boldsymbol{a}}(p+q) \quad \text{is CLC.}$$

So we just need to show that $p + q$ is log-concave in the positive orthant. For any $\boldsymbol{a} \in \mathbb{R}^n_{>0}$, define $Q_1 := \nabla^2 p(\boldsymbol{a})$ and $Q_2 := \nabla^2 q(\boldsymbol{a})$ to get

$$(Q_1 \boldsymbol{b})_j = \sum_i b_i \partial_{x_i} \partial_{x_j} p(\boldsymbol{a}) = \partial_{x_j} \nabla_{\boldsymbol{b}} p(\boldsymbol{a}) = \partial_{x_j} \nabla_{\boldsymbol{c}} q(\boldsymbol{a}) = (Q_2 \boldsymbol{c})_j.$$

That is $Q_1 \boldsymbol{b} = Q_2 \boldsymbol{c} \neq \boldsymbol{0}$. Log-concavity of p, q then implies Q_1 and Q_2 are both NSD on $(Q_1 \boldsymbol{b})^\perp = (Q_2 \boldsymbol{c})^\perp$ by the Lorentz equivalence.

Therefore $Q_1 + Q_2 = \nabla^2[p+q](\boldsymbol{a})$ is NSD on this $(n-1)$-dimensional subspace, which implies $\nabla^2 \log[p+q](\boldsymbol{a})$ is NSD by Lorentz equivalence.

Properties of CLC polynomials, revisited

Proposition

If $p, q \in \mathbb{R}_+[\boldsymbol{x}]$ are CLC polynomials, then

1. $\nabla_{\boldsymbol{a}} p$ for $\boldsymbol{a} \in \mathbb{R}_+^n$ and $p|_{x_i=0}$ are CLC.
2. The Hessian $\nabla^2 p(\boldsymbol{a})$ is Lorentz for all $\boldsymbol{a} \in \mathbb{R}_+^n$.
3. $p(A\boldsymbol{x})$ is CLC for all $n \times m$ matrices A with non-negative entries.
4. $p(\boldsymbol{a} \cdot t + \boldsymbol{b} \cdot s) \in \mathbb{R}_+[t, s]$ is CLC for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}_+^n$.
5. $p(\boldsymbol{x}) \cdot q(\boldsymbol{z}) \in \mathbb{R}_+[\boldsymbol{x}, \boldsymbol{z}]$ is CLC.
6. $p(\boldsymbol{x}) \cdot q(\boldsymbol{x}) \in \mathbb{R}_+[\boldsymbol{x}]$ is CLC.

Lemma (sum-of-CLCs)

If $p, q \in \mathbb{R}_+[\boldsymbol{x}]$ are d-homogeneous CLC polynomials such that $\nabla_{\boldsymbol{b}} p = \nabla_{\boldsymbol{c}} q \not\equiv 0$ for some $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}_+^n$, then $p + q$ is CLC.

Two main corollaries: Reduction to quadratics and symbol theorem.

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Reduction to quadratics

Theorem (Anari-Oveis Gharan-Vinzant '19; see also Brändén-Huh '19)

A d-homogeneous polynomial $p \in \mathbb{R}_+[\boldsymbol{x}]$ is CLC iff:

1. *For all $\boldsymbol{\mu} \in \mathbb{Z}_+^n$ with $|\boldsymbol{\mu}| \leq d-2$, $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} p$ is indecomposable.*
2. *For all $\boldsymbol{\mu} \in \mathbb{Z}_+^n$ with $|\boldsymbol{\mu}| = d-2$, $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} p$ is log-concave in $\mathbb{R}_+^n$.*

Indecomposable polynomial: p cannot be written as $p = f + g$ where $f, g \not\equiv 0$ depend on disjoint variables.

Easy direction ($\Longrightarrow$): If $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} p$ is decomposable and of degree d', then

$$\nabla_{\mathbf{1}}^{d'-2}(\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} p) = \nabla_{\mathbf{1}}^{d'-2}(f+g) = \nabla_{\mathbf{1}}^{d'-2} f + \nabla_{\mathbf{1}}^{d'-2} g$$

is a decomposable quadratic form. Therefore $p(\boldsymbol{x}) = \boldsymbol{x}^\top \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \boldsymbol{x}$, where A, B are Lorentz matrices since f and g are CLC (plug in 0).

Contradiction: Hessian of p has two positive eigenvalues.

Proof of reduction to quadratics, harder direction

Simplify: Assume that $p_{\boldsymbol{\mu}} > 0$ for all $|\boldsymbol{\mu}| = d$. $\implies$ Stronger than indecomposable. (One can limit the positive coefficients case to the general indecomposable case, but this is **not obvious** [Brändén-Huh '19].)

Lemma: If $\partial_{x_i} p$ is CLC for all i, then $\nabla_{\boldsymbol{a}} p$ is CLC for all $\boldsymbol{a} \in \mathbb{R}^n_+$.
Proof: First assume $\boldsymbol{a} > \mathbf{0}$, and let $D_k := \sum_{i=1}^k a_i \partial_{x_i}$. Assume by induction that $D_k p$ is CLC. By the sum-of-CLCs lemms, we have that

$$a_{k+1}\partial_{x_{k+1}}(D_k p) = D_k(a_{k+1}\partial_{x_{k+1}} p)$$
$$\implies \quad D_{k+1} p = (D_k + a_{k+1}\partial_{x_{k+1}})p \text{ is CLC.}$$

Note that sum-of-CLCs applies because $\partial_{x_{k+1}}(D_k p) \not\equiv 0$, since $p_{\boldsymbol{\mu}} > 0$. For $\boldsymbol{a} \in \mathbb{R}^n_+$ we simply skip the entries of $\boldsymbol{a}$ which are 0.

Including indecomposability: Need to order the variables in such a way so that $\partial_{x_{k+1}}(D_k p) \not\equiv 0$. (Exercise.)

Proof of reduction to quadratics, harder direction

Theorem (Anari-Oveis Gharan-Vinzant '19; see also Brändén-Huh '19)

A d-homogeneous polynomial $p \in \mathbb{R}_+[\boldsymbol{x}]$ *is CLC iff:*

1. *For all* $\boldsymbol{\mu} \in \mathbb{Z}_+^n$ *with* $|\boldsymbol{\mu}| \leq d-2$, $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} p$ *is indecomposable.*
2. *For all* $\boldsymbol{\mu} \in \mathbb{Z}_+^n$ *with* $|\boldsymbol{\mu}| = d-2$, $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} p$ *is log-concave in* $\mathbb{R}_+^n$.

Assume: $p_{\boldsymbol{\mu}} > 0$ for all $|\boldsymbol{\mu}| = d$.
Lemma: If $\partial_{x_i} p$ is CLC for all i, then $\nabla_{\boldsymbol{a}} p$ is CLC for all $\boldsymbol{a} \in \mathbb{R}_+^n$.

Other direction ($\Longleftarrow$): It suffices to show that p is log-concave in $\mathbb{R}_+^n$ and that $\nabla_{\boldsymbol{a}} p$ is CLC for all $\boldsymbol{a} \in \mathbb{R}_+^n$. By induction on degree, $\partial_{x_i} p$ is CLC for all i. Thus the lemma applies, and $\nabla_{\boldsymbol{a}} p$ is CLC for all $\boldsymbol{a} \in \mathbb{R}_+^n$.

The log-concavity of p then follows from the fact that p is log-concave at $\boldsymbol{a}$ iff $\nabla_{\boldsymbol{a}} p$ is log-concave at $\boldsymbol{a}$. **Why?** Lorentz equiv. and Euler's identity:

$$\nabla^2[\nabla_{\boldsymbol{a}} p](\boldsymbol{a}) = \left[\textstyle\sum_i a_i \partial_{x_j} \partial_{x_k} \partial_{x_i} p(\boldsymbol{a})\right]_{j,k=1}^n = \left[\partial_{x_j} \partial_{x_k} p(\boldsymbol{a})\right]_{j,k=1}^n = \nabla^2 p(\boldsymbol{a}).$$

Corollaries of the reduction to quadratics: real stability

Fact: If the quadratic form $p(\boldsymbol{x}) := \boldsymbol{x}^\top A \boldsymbol{x} \in \mathbb{R}_+^n[\boldsymbol{x}]$ is real stable, then A is Lorentz. (Note that A is the constant Hessian of p in this case.)
Corollary: Homogeneous real stable polynomials are CLC.

Proof for quadratics: By Perron-Frobenius, A has an eigenvalue $\lambda_1 > 0$ with corresponding eigenvector $\boldsymbol{a}$ which has non-negative entries. Suppose A has a second positive eigenvalue λ_2 with corresponding eigenvector $\boldsymbol{b}$.

Contradiction: This implies the linear restriction $p(\boldsymbol{a} \cdot t + \boldsymbol{b})$ has no zeros.

Note: Perron-Frobenius usually requires strictly positive entries. However, indecomposability implies Perron-Frobenius can be used. **This is one possible intuition for indecomposability.**

Note: The converse is also true: A quadratic is real stable if and only if the associated matrix is Lorentz. Actually, many equivalences at the level of quadratics. **Proof:** Exercise.

Corollaries of the quadratic reduction: ULC

Fact: Homogeneous $p \in \mathbb{R}_+[x_1, x_2]$ is CLC iff its coefficients are ULC.

Proof: Follows directly from the reduction to quadratics:

1. **Indecomposable:** Equivalent to having no internal zeros in coefficient sequence. (Take derivatives until $ax_1^k + bx_2^j$ with $|k - j| \geq 2$ and $a, b \neq 0$.)
2. **Quadratic derivatives:** Each derivative of degree $d - 2$ picks out a sequence of 3 coefficients in p. We just need to show that these quadratics have non-negative discriminant to prove ULC. A bivariate quadratic form looks like:

$$\boldsymbol{x}^\top \begin{bmatrix} a & b \\ b & c \end{bmatrix} \boldsymbol{x} = ax_1^2 + 2bx_1x_2 + cx_2^2$$

This matrix is Lorentz iff $\det \leq 0$ iff $ac \leq b^2$ iff $(2b)^2 - 4ac \geq 0$.

Outline

1. Completely log-concave (Lorentzian) polynomials
   - Motivation
   - Definition
2. Properties of completely log-concave polynomials
   - Log-concavity and Lorentz signature
   - Products of CLC polynomials
3. Reduction to quadratics
   - Proof of the reduction
   - Corollaries of the reduction
4. Symbol theorem for CLC polynomials

Symbol theorem for multiaffine CLC polynomials

Definition: The **symbol** of a linear operator $T : \mathbb{R}_+^{\mathbf{1}}[\boldsymbol{x}] \to \mathbb{R}_+[\boldsymbol{x}]$:

$$\text{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x}, \boldsymbol{z}) := T\left[\prod_{i=1}^{n}(x_i + z_i)\right] = \sum_{\boldsymbol{\mu} \leq \mathbf{1}} \boldsymbol{z}^{\mathbf{1}-\boldsymbol{\mu}} T[\boldsymbol{x}^{\boldsymbol{\mu}}]$$

Here T acts only on $\boldsymbol{x}$ and $\boldsymbol{\mu} \leq \mathbf{1}$ is entrywise.

Theorem (Anari-Liu-Oveis Gharan-Vinzant '19, Brändén-Huh '19)

For a given linear operator $T : \mathbb{R}_+^{\mathbf{1}}[\boldsymbol{x}] \to \mathbb{R}_+[\boldsymbol{x}]$, we have that T preserves CLC (allowing $\equiv 0$) if $\text{Symb}^{\mathbf{1}}[T](\boldsymbol{x}, \boldsymbol{z})$ is CLC.

Proof: $T[p](\boldsymbol{x}) = \prod_{i=1}^{n} (\partial_{z_i} + \partial_{t_i})\big|_{z_i=t_i=0} \left[\text{Symb}^{\mathbf{1}}[T](\boldsymbol{x}, \boldsymbol{z}) \cdot p(\boldsymbol{t})\right].$

Corollary: Homogeneous real stability preservers preserve CLC.

Fact: Polarization also preserves CLC. $\implies$ More general symbol theorem follows from the same polarization techniques as in the real stable case.