Completely Log-concave (Lorentzian) Polynomials Exercises

Jonathan Leake

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Definition. A matrix Q is said to be **Lorentz** if it is a Hermitian matrix in the close of the set of all matrices with Lorentz signature $(+,-,-,\dots,-)$.

Definition. A polynomial $p \in \mathbb{R}_+[\boldsymbol{x}]$ is said to be **indecomposable** if there is no way to write $p = f + g$ where $f, g \not\equiv 0$ depend on disjoint sets of variables.

Definition. A d-homogeneous polynomial $p \in \mathbb{R}_+[\boldsymbol{x}]$ is said to be **completely log-concave** if for all $k \in \mathbb{Z}_+$ and all choices of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k \in \mathbb{R}^n_+$, we have that

$$\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} p = \left(\prod_{i=1}^{k} \sum_{j=1}^{n} v_{ij} \right) p$$

is log-concave in the positive orthant. We also consider the zero polynomial to be completely log-concave.

Exercises

1. Complete the proof of the fact that a d-homogeneous polynomial $p \in \mathbb{R}_+[\boldsymbol{x}]$ is completely log-concave if and only if the following hold:

   (a) For all $\boldsymbol{\mu} \in \mathbb{Z}^n_+$ with $|\boldsymbol{\mu}| \leq d-2$, we have that $\partial^{\boldsymbol{\mu}}_{\boldsymbol{x}} p$ is indecomposable.

   (b) For all $\boldsymbol{\mu} \in \mathbb{Z}^n_+$ with $|\boldsymbol{\mu}| = d-2$, we have $\partial^{\boldsymbol{\mu}}_{\boldsymbol{x}} p = \boldsymbol{x}^\top Q \boldsymbol{x}$ is such that Q is Lorentz.

   Recall that we proved this in the case that $p_{\boldsymbol{\mu}} > 0$ for all $|\boldsymbol{\mu}| = d$, and so all that's left to be done is the ($\Longleftarrow$) direction under the assumption of indecomposability. (**Hint:** Need to order the variables in a special way.)

2. Prove that if $\boldsymbol{x}^\top Q \boldsymbol{x}$ is indecomposable, then the Perron-Frobenius theorem holds for Q.

3. Let $p(\boldsymbol{x}) = \boldsymbol{x}^\top Q \boldsymbol{x}$ where Q is a real symmetric matrix with non-negative entries. Prove that the following are equivalent.

   (a) Q is Lorentz.

   (b) p is real stable.

   (c) $\boldsymbol{v}^\top Q \boldsymbol{w} \geq \sqrt{\boldsymbol{v}^\top Q \boldsymbol{v} \cdot \boldsymbol{w}^\top Q \boldsymbol{w}}$ for all $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n_+$.

   (d) $\sqrt{p}$ is concave in $\mathbb{R}^n_+$.

   (e) p is log-concave in $\mathbb{R}^n_+$.

   (f) The Hessian of $\log p$ is negative semidefinite at all points in $\mathbb{R}^n_{>0}$.

   (g) The Hessian of $\log p$ is negative semidefinite at some point in $\mathbb{R}^n_{>0}$.

4. Given a d-homogeneous polynomial $p \in \mathbb{R}_+[\boldsymbol{x}]$, prove that the following are equivalent:

(a) **Completely log-concave:** p is completely log-concave.

(b) **Coordinate derivatives:** For all $k \in \mathbb{Z}_+$ and all choices of $i_1, \ldots, i_k \in [n]$, we have that $\partial_{x_{i_1}} \cdots \partial_{x_{i_k}} p$ is log-concave in $\mathbb{R}^n_+$.

(c) **Coordinate quadratics plus one:** For all choices of $D_1, \ldots, D_{d-2} \in \{\partial_{x_1}, \ldots, \partial_{x_n}, \nabla_{\mathbf{1}}\}$, we have that $D_1 D_2 \cdots D_{d-2} p$ is log-concave in $\mathbb{R}^n_+$.

(d) **Positive orthant quadratics:** For all choices of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{d-2} \in \mathbb{R}^n_+$, we have that $\nabla_{\boldsymbol{v}_1} \nabla_{\boldsymbol{v}_2} \cdots \nabla_{\boldsymbol{v}_{d-2}} p$ is log-concave in $\mathbb{R}^n_+$.

(e) **Alexandrov-Fenchel inequalities:** For all $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d \in \mathbb{R}^n_+$, we have that

$$\nabla_{\boldsymbol{v}_1} \nabla_{\boldsymbol{v}_2} \nabla_{\boldsymbol{v}_3} \cdots \nabla_{\boldsymbol{v}_d} p \geq \sqrt{(\nabla_{\boldsymbol{v}_1} \nabla_{\boldsymbol{v}_1} \nabla_{\boldsymbol{v}_3} \cdots \nabla_{\boldsymbol{v}_d} p) \cdot (\nabla_{\boldsymbol{v}_2} \nabla_{\boldsymbol{v}_2} \nabla_{\boldsymbol{v}_3} \cdots \nabla_{\boldsymbol{v}_d} p)}.$$

5. Prove that all homogeneous real stable polynomials in $\mathbb{R}_+[\boldsymbol{x}]$ are completely log-concave.

6. Given a real symmetric $n \times n$ matrix Q with non-negative entries, prove that Q is Lorentz if and only if

$$(-1)^{|I|} \det(Q_{I,I}) \leq 0$$

for all $I \subseteq [n]$, where $\det(Q_{I,I})$ denotes the principal minor corresponding to the rows and columns indexed by I.

7. Construct an algorithm for checking if p is completely log-concave, assuming you know the coefficients of p exactly. Is there an algorithm for checking if p is real stable? (**Hint:** For the real stable question, consider the strong Rayleigh inequalities.)

8. State and prove a version of exercise (3) in the case that Q is positive semidefinite. Is there an interesting polynomial theory that comes out of this observation? (This is more or less a "conceptual open problem", and you should let me know if you have thoughts on possible applications.)

9. **Open problem:** Is there a similar polynomial theory to that of completely log-concave (Lorentzian) polynomials which allows for "close-to-Lorentz" matrices? (Here "close-to-Lorentz" should mean that the second largest eigenvalue is allowed to be positive, but small.)

10. **Analogue to the strong Rayleigh inequalities:** Let $p \in \mathbb{R}_+[\boldsymbol{x}]$ be a multiaffine d-homogeneous completely log-concave polynomial. Show that for all $i, j \in [n]$ and all $\boldsymbol{x} \in \mathbb{R}^n_+$, we have that

$$\partial_{x_i} p(\boldsymbol{x}) \cdot \partial_{x_j} p(\boldsymbol{x}) - 2\left(1 - \frac{1}{d}\right) \cdot p(\boldsymbol{x}) \cdot \partial_{x_i} \partial_{x_j} p(\boldsymbol{x}) \geq 0.$$

11. Let $p \in \mathbb{R}_+[\boldsymbol{x}]$ be a multiaffine d-homogeneous polynomial. Prove that p is completely log-concave if and only if p is log-concave in the positive orthant. (**Hint:** What is $\lim_{t \to +\infty} \frac{1}{t} \cdot p(t, x_2, \ldots, x_n)$ equal to?)

12. Let $p \in \mathbb{R}_+[\boldsymbol{x}]$ be a polynomial of total degree at most d which satisfies the definition of completely log-concave polynomial, except that it is not homogeneous. Prove that the d-homogenization of p is not necessarily completely log-concave. On the other hand, show that the homogenization of any real stable polynomial $p \in \mathbb{R}_+[\boldsymbol{x}]$ is also real stable. (**Hint:** For the real stable part, the non-negativity of the coefficients is necessary, and therefore one cannot hope to use the Borcea-Brändén characterization here.)

13. Let $p \in \mathbb{R}_+[\boldsymbol{x}]$ be a polynomial of total degree at most d which satisfies the definition of completely log-concave polynomial, except that it is not homogeneous. Write the d-homogenization of p as

$$\mathrm{Hmg}(p) = \sum_{i=0}^{d} x^i p_i(\boldsymbol{x}),$$

where p_i is homogeneous of degree $d - i$. Prove that

$$P(\boldsymbol{x}) := \sum_{i=0}^{d} \frac{x^i}{i!} p_i(\boldsymbol{x})$$

is a homogeneous completely log-concave polynomial. Prove that this property actually characterizes "non-homogenous completely log-concave" polynomials.

14. Prove that the polarization operator used for real stable polynomials preserves complete log-concavity. (**Hint:** How does the derivative ∂_{x_i} commute with polarization?)

15. **Symbol theorem:** Given a linear operator $T : \mathbb{R}_+^{\boldsymbol{\lambda}}[\boldsymbol{x}] \to \mathbb{R}_+[\boldsymbol{x}]$, define the **symbol** of T via

$$\mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x}, \boldsymbol{z}) := T\left[\prod_{i=1}^{n} (x_i + z_i)^{\lambda_i}\right] = \sum_{\boldsymbol{\mu} \leq \boldsymbol{\lambda}} \binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}} \boldsymbol{z}^{\boldsymbol{\lambda} - \boldsymbol{\mu}} T[\boldsymbol{x}^{\boldsymbol{\mu}}],$$

where $\binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}}$ is the product of binomial coefficients. Prove that if $\mathrm{Symb}^{\boldsymbol{\lambda}}[T]$ is completely log-concave, then T preserves the space of completely log-concave polynomials. (**Hint:** Use the polarization.)

16. Given a d-homogeneous $p \in \mathbb{R}_+[\boldsymbol{x}]$, define the following **normalization** operator:

$$N[p] := \sum_{\boldsymbol{\mu}} \binom{d}{\boldsymbol{\mu}} p_{\boldsymbol{\mu}},$$

where $\binom{d}{\boldsymbol{\mu}}$ is the multinomial coefficient. We say that p is **denormalized Lorentzian** whenever $N[p]$ is Lorentzian. Prove that the product of two denormalized Lorentzian polynomials is denormalized Lorentzian. (**Hint:** Use the symbol theorem of the previous exercise. Note that this somewhat requires the equivalent definition of **Lorentzian** polynomials, due to Brändén-Huh, which replaces indecomposability with matroidal/M-convex support. If you are not familiar with this, don't worry: we'll talk about it next week.)