Applications of CLC/Lorentzian Polynomials

Polynomial Capacity: Theory, Applications, Generalizations

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Notation

Polynomial notation:

- $\mathbb{R}, \mathbb{R}_+, \mathbb{C}, \mathbb{Z}_+ :=$ reals, non-negative reals, complex numbers, non-negative integers.
- $\boldsymbol{x}^{\boldsymbol{\mu}} := \prod_i x_i^{\mu_i}$ and $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ is entrywise.
- $\mathbb{R}[\boldsymbol{x}] :=$ v.s. of real polynomials in n variables.
- $\mathbb{R}_+[\boldsymbol{x}] :=$ v.s. of real polynomials with non-negative coefficients.
- $\mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}] :=$ v.s. of polynomials of degree at most λ_i in x_i.
- For $p \in \mathbb{R}[\boldsymbol{x}]$, we write $p(\boldsymbol{x}) = \sum_{\boldsymbol{\mu}} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$.
- For d-homogeneous $p \in \mathbb{R}[\boldsymbol{x}]$, we write $p(\boldsymbol{x}) = \sum_{|\boldsymbol{\mu}|=d} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$.
- The **support** of p is the set of $\boldsymbol{\mu} \in \mathbb{Z}_+^n$ for which $p_{\boldsymbol{\mu}} \neq 0$.
- $\frac{d}{dx} = \frac{\partial}{\partial x} = \partial_x :=$ derivative with respect to x, and $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} := \prod_i \partial_{x_i}^{\mu_i}$.
- $p(\boldsymbol{a} \cdot t + \boldsymbol{b}) = p(a_1 t + b_1, \ldots, a_n t + b_n) \in \mathbb{R}^{\lambda_1 + \cdots + \lambda_n}[t]$ is a **linear restriction** of the polynomial $p \in \mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$, where $\boldsymbol{a} \in \mathbb{R}_+^n$ and $\boldsymbol{b} \in \mathbb{R}^n$.

Outline

1. Connection to matroids
   - Properties and examples of matroids
   - Basis exchange graph
   - Basis generating polynomials are CLC
   - Lorentzian polynomials
2. Mason's strongest conjecture
   - The independent set polynomial
   - Independent set polynomials are CLC
3. Sampling bases of a matroid
   - Sampling via random walks
   - Mixing time via λ_2 (second largest eigenvalue)
   - Local-to-global theorem for λ_2
   - Basis sampling overview
4. Foreshadowing: Counting bases in the intersection of two matroids

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Matroids

Matroid: $M = (E, \mathcal{I})$ where E is the **ground set** and $\mathcal{I} \subseteq 2^E$ are the **independent** subsets, which satisfy:

1. **Nonempty:** $\mathcal{I} \neq \varnothing$.
2. **Hereditary:** $B \in \mathcal{I}$ and $A \subseteq B$ implies $A \in \mathcal{I}$.
3. **Exchange/Augmentation:** For all $A, B \in \mathcal{I}$ such that $|A| < |B|$, there exists $e \in B \setminus A$ such that $A \cup \{e\} \in \mathcal{I}$.

E.g.: A set of vectors in a vector space, with $\mathcal{I}$ given by linearly independent subsets (**linear matroid**). The set of edges of a graph, with $\mathcal{I}$ given by subsets with no cycles (**graphic matroid**). **Many more...**

Maximal $B \in \mathcal{I}$ are the **bases**, $\mathcal{B} \subset \mathcal{I}$, of M. **Another definition of M:**

3. **Exchange:** For any bases $B_1, B_2 \in \mathcal{B}$ and any $e_1 \in B_1 \setminus B_2$, there exists $e_2 \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}$.

The spanning tree polynomial is a basis-generating polynomial. **Others?**

Examples of matroids

Linear matroid example: $E = \{(1,0,0),(1,1,0),(0,1,0),(1,1,1)\} \subset \mathbb{R}^3$, with $\mathcal{I}$ given by linearly independent subsets.

Then $\mathcal{I}$ consists of the empty set, all one-element subsets, all two-elements subsets, and all three-element subsets containing $(1,1,1)$.

The **bases** of $M = (E, \mathcal{I})$ are the three-element sets in $\mathcal{I}$; they are precisely the linear bases of $\mathbb{R}^3$. The **rank** of M is therefore 3.

Uniform matroid example: Let E be any set, and let $\mathcal{I}$ be the set of all subsets of size at most d (for any d).

The bases of M are the d-element sets in $\mathcal{I}$, and the **rank** of M is d.

The basis generating polynomial is the elementary symmetric polynomial, which is real stable. **However, not all basis generating polynomials are real stable.**

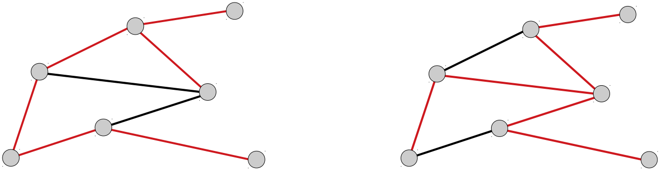
Basis exchange for linear/graphic matroids

Matroid: $M = (E, \mathcal{B})$ where E is the **ground set** and $\mathcal{B} \subset 2^E$ are the **bases** of M. They are all of the same size, satisfying:

- **Basis exchange:** For any bases $B_1, B_2 \in \mathcal{B}$ and any $e_1 \in B_1 \setminus B_2$, there exists $e_2 \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}$.

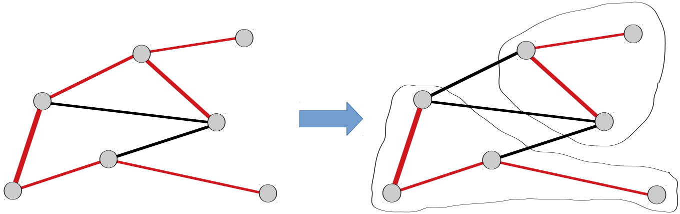
Linear matroids: Let B_1, B_2 be two different bases of $\mathbb{C}^n$, and fix some $e_1 \in B_1 \setminus B_2$. Then $B_1 \setminus \{e_1\}$ spans V. Since B_2 is a basis, there is some $e_2 \in B_2 \setminus V$. With this, $(B_1 \setminus \{e_1\}) \cup \{e_2\}$ is another basis of $\mathbb{C}^n$.

Graphic matroids: Spanning trees of a connected graph.



Basis exchange: For any bases $B_1, B_2 \in \mathcal{B}$ and any $e_1 \in B_1 \setminus B_2$, there exists $e_2 \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}$.

Graphic matroids: Let B_1, B_2 be two different spanning trees of a connected graph G, and fix some $e_1 \in B_1 \setminus B_2$.



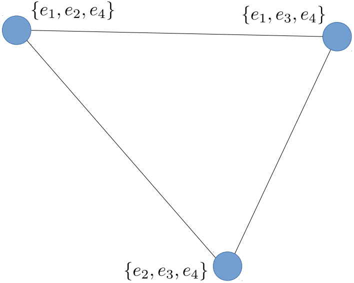
1. Remove e_1 from B_1 to partition the vertices $V = U \sqcup W$ based on which vertices are connected by edges in $B_1 \setminus \{e_1\}$ (pictured).
2. Pick an edge $e_2 \in B_2$ which connects U and W, and add it B_1.
3. The set $(B_1 \setminus \{e_1\}) \cup \{e_2\}$ must be a spanning tree, since it has no cycles and connects all vertices.

Basis exchange graph

Given **any** matroid, $M = (E, \mathcal{B})$, construct a graph with the bases as vertices. Let two bases be connected by an edge if there is an exchange to go from one to the other. (That is, if $|B_1 \setminus B_2| = 1 \iff |B_1 \Delta B_2| = 2$.)

Fact: The basis exchange graph of any matroid is connected.

Example: $E = \{(1,0,0),(1,1,0),(0,1,0),(1,1,1)\} = \{e_1, e_2, e_3, e_4\}$.



Uniform matroid: Regular, highly symmetric graph.

Basis exchange graph

Given **any** matroid, $M = (E, \mathcal{B})$, construct a graph with the bases as vertices. Let two bases be connected by an edge if there is an exchange to go from one to the other. (That is, if $|B_1 \setminus B_2| = 1 \iff |B_1 \Delta B_2| = 2$.)

Fact: The basis exchange graph of any matroid is connected.

Proof: For $B_0, B' \in \mathcal{B}$, pick any $e_1 \in B_0 \setminus B'$ and move along an edge to $B_1 := (B_0 \setminus \{e_1\}) \cup \{e_2\}$ for some $e_2 \in B' \setminus B_0$. We are guaranteed that $|B_1 \setminus B'| < |B_0 \setminus B'| < \infty$ in this case. By continuing this process, we eventually have $|B_k \setminus B'| = 0$. Since $|B_k| = |B'|$, we in fact have $B_k = B'$.

Corollary: If $|B| \geq 2$ and every $e \in E$ is included in some basis, then there is no non-trivial partition $E := F \sqcup G$ such that: for all $B \in \mathcal{B}$ either $B \subseteq F$ or $B \subseteq G$.

Proof: Suppose such a partition exists. By assumption, there are bases $B_1 \subseteq F$ and $B_2 \subseteq G$. Therefore, $|B_1 \setminus B_2| = |B_1| \geq 2$. Since this is true of all such bases, there is no way to move from B_1 to B_2 via exchanges.

Completely log-concave (CLC) polynomials

Definition (Gurvits '09, Anari-Oveis Gharan-Vinzant '19)

A d-homogeneous polynomial $p \in \mathbb{R}_+[\boldsymbol{x}]$ is **completely log-concave (CLC)** if for any choice of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}_+^n$ for any k, we have that

$$\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} p := \left(\sum_i v_{1i} \partial_{x_i}\right) \cdots \left(\sum_i v_{ki} \partial_{x_i}\right) p$$

is log-concave in the positive orthant or $\equiv 0$.

Theorem (Anari-Oveis Gharan-Vinzant '19; see also Brändén-Huh '19)

A d-homogeneous polynomial $p \in \mathbb{R}_+[\boldsymbol{x}]$ is CLC iff:

1. *For all $\boldsymbol{\mu} \in \mathbb{Z}_+^n$ with $|\boldsymbol{\mu}| \leq d-2$, $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} p$ is indecomposable.*
2. *For all $\boldsymbol{\mu} \in \mathbb{Z}_+^n$ with $|\boldsymbol{\mu}| = d-2$, $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} p = \boldsymbol{x}^\top Q \boldsymbol{x}$ with Q Lorentz.*

Indecomposable polynomial: p cannot be written as $p = f + g$ where $f, g \not\equiv 0$ depend on disjoint variables. **Condition on the support of p.**
Lorentz matrix: Signature $(+, -, -, \ldots, -)$, or in the closure.

Basis generating polynomials

Corollary: If $|B| \geq 2$ and every $e \in E$ is included in some basis, then there is no non-trivial partition $E := F \sqcup G$ such that: for all $B \in \mathcal{B}$ either $B \subseteq F$ or $B \subseteq G$.

Basis generating polynomial: Given a matroid $M = (E, \mathcal{B})$, we define

$$p_M(\boldsymbol{x}) := \sum_{B \in \mathcal{B}} \boldsymbol{x}^B = \sum_{B \in \mathcal{B}} \prod_{e \in B} x_e \quad \in \quad \mathbb{R}^{\mathbf{1}}[\boldsymbol{x}] = \mathbb{R}^{\mathbf{1}}[(x_e)_{e \in E}].$$

Indecomposable polynomial: p cannot be written as $p = f + g$ where $f, g \not\equiv 0$ depend on disjoint variables.

Corollary: Every basis generating polynomial is indecomposable.

More: $p_{M/e}(\boldsymbol{x}) = \partial_{x_e} p_M(\boldsymbol{x})$, where M/e denotes **matroid contraction**, where one keeps all bases in $\mathcal{B}$ that contain e (and then remove e from all).

This is another matroid, so $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} p_M$ is indecomposable for all $|\boldsymbol{\mu}| \leq d - 2$.

Basis generating polynomials

Basis generating polynomial: Given a matroid $M = (E, \mathcal{B})$, we define

$$p_M(\boldsymbol{x}) := \sum_{B \in \mathcal{B}} \boldsymbol{x}^B = \sum_{B \in \mathcal{B}} \prod_{e \in B} x_e \quad \in \quad \mathbb{R}^{\mathbf{1}}[\boldsymbol{x}] = \mathbb{R}^{\mathbf{1}}[(x_e)_{e \in E}].$$

Last slide: $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} p_M$ is indecomposable for all $|\boldsymbol{\mu}| \leq d - 2$.

Now: For $|\boldsymbol{\mu}| = d - 2$, we have $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} p$ is the basis-generating polynomial of a **rank-two** matroid. **Fact:** The associated quadratic form is Lorentz.

Proof: Remove all $e \in E$ which are outside of all bases. The following is then an equivalence relation:

$$e \sim f \text{ for } e, f \in E \quad \iff \quad \{e, f\} \notin \mathcal{B}.$$

To see this, we just need to show transitivity. Suppose $e \sim f$ and $f \sim g$, but $\{e, g\} \in \mathcal{B}$. Pick $\{f, h\} \in \mathcal{B}$ and try to do basis exchange from $B_1 = \{f, h\}$ to $B_2 = \{e, g\}$ after removing $e_1 = h$. This forces either $\{e, f\}$ or $\{f, g\}$ to be a basis in $\mathcal{B}$.

Basis generating polynomials

Fact: The quadratic form associated to a rank-two matroid is Lorentz.

Proof: Remove all $e \in E$ which are outside of all bases. The following is then an equivalence relation:

$$e \sim f \text{ for } e, f \in E \quad \iff \quad \{e, f\} \notin \mathcal{B}.$$

Let $E = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_m$ be the equivalence classes of E. We can write the basis generating polynomial as

$$2 \cdot p_M(\boldsymbol{x}) = 2 \cdot \sum_{B \in \mathcal{B}} \boldsymbol{x}^B = \boldsymbol{x}^\top \left(\mathbf{1}_E \mathbf{1}_E^\top - \sum_{i=1}^m \mathbf{1}_{S_i} \mathbf{1}_{S_i}^\top \right) \boldsymbol{x} =: \boldsymbol{x}^\top Q \boldsymbol{x}.$$

Subtracting a PSD matrix can only decrease eigenvalues, and Q is real symmetric with non-negative entries. Therefore Q is Lorentz.

Corollary: Every matroid basis generating polynomial is CLC.

Equivalent theory: Lorentzian polynomials

Every matroid basis generating polynomial is CLC. **A sort of converse to this is also true**; [Brändén-Huh '19] calls such polynomials **Lorentzian**.

Theorem (Anari-Oveis Gharan-Vinzant '19)

A d-homogeneous polynomial $p \in \mathbb{R}_+[\mathbf{x}]$ is CLC iff:

1. *For all $\boldsymbol{\mu} \in \mathbb{Z}_+^n$ with $|\boldsymbol{\mu}| \leq d-2$, $\partial_{\mathbf{x}}^{\boldsymbol{\mu}} p$ is indecomposable.*
2. *For all $\boldsymbol{\mu} \in \mathbb{Z}_+^n$ with $|\boldsymbol{\mu}| = d-2$, $\partial_{\mathbf{x}}^{\boldsymbol{\mu}} p = \mathbf{x}^\top Q \mathbf{x}$ with Q Lorentz.*

Theorem (Brändén-Huh '19; definition of Lorentzian polynomial)

*A d-homogeneous **multiaffine** polynomial $p \in \mathbb{R}_+[\mathbf{x}]$ is CLC iff:*

1. *The **support** of p is the set of bases of a matroid.*
2. *For all $\boldsymbol{\mu} \in \mathbb{Z}_+^n$ with $|\boldsymbol{\mu}| = d-2$, $\partial_{\mathbf{x}}^{\boldsymbol{\mu}} p = \mathbf{x}^\top Q \mathbf{x}$ with Q Lorentz.*

For non-multiaffine: Replace "the set of bases of a matroid" with "M-convex". $\implies$ Natural generalization of matroid to "higher degree".

Outline

Mason's strongest conjecture

Conjecture [Mason '75]: If $M = (E, \mathcal{I})$ is a matroid such that $|E| = n$, and I_k denotes the number of independent sets of M of size k, then $(I_k)_{k=0}^n$ forms an ultra log-concave sequence **with respect to** n.

Weaker Mason's conjecture: Log-concavity [Adiprasito-Huh-Katz '15].

Independent set generating polynomial for $M = (E, \mathcal{I})$ with $|E| = n$:

$$q_M(\boldsymbol{x}, y) := \sum_{I \in \mathcal{I}} \boldsymbol{x}^I y^{n-|I|} = \sum_{I \in \mathcal{I}} y^{n-|I|} \prod_{e \in I} x_e = y^{n-r} \cdot \sum_{I \in \mathcal{I}} \boldsymbol{x}^I y^{r-|I|}$$

where r is the rank of the matroid M. (y^{n-r} factor is **crucial**.)

Theorem [Anari-Liu-Oveis Gharan-Vinzant '19, Brändén-Huh '19]: For any matroid the polynomial q_M is CLC/Lorentzian.

Corollary: Mason's strongest conjecture holds.

Proof: $q_M(t, t, \ldots, t, s) = \sum_{k=0}^{n} I_k t^k s^{n-k}$ is CLC $\iff$ ULC coefficients.

Proof that the independent set polynomial is CLC

Independent set generating polynomial for $M = (E, \mathcal{I})$ with $|E| = n$:

$$q_M(\boldsymbol{x}, y) := \sum_{I \in \mathcal{I}} \boldsymbol{x}^I y^{n-|I|} = \sum_{I \in \mathcal{I}} y^{n-|I|} \prod_{e \in I} x_e.$$

What does ∂_{x_e} do? Matroid contraction.
What about ∂_y^k? Matroid truncation: $\mathcal{I}_{n-k} := \{I \in \mathcal{I} \;:\; |I| \leq n-k\}$.
Easy to verify the matroid axioms in terms of the independent sets.

For all $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} \partial_y^k$ such that $|\mu| + k \leq n-2$, the polynomial $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} \partial_y^k q_M$ is a **tweaked version** of independent set generating polynomial of a matroid.

Support does not depend on value of coefficients $\implies$ indecomposable.

Leaves one thing to check, by induction: Given any matroid M on $|E| = n \geq 2$ elements, need to show that

$$\partial_y^{n-2} q_M(\boldsymbol{x}, y) = \begin{bmatrix} \boldsymbol{x} \\ y \end{bmatrix}^\top Q \begin{bmatrix} \boldsymbol{x} \\ y \end{bmatrix}$$

is such that Q is Lorentz.

Proof that the independent set polynomial is CLC

Given any matroid M on $|E| = n \geq 2$ elements, need to show that

$$\partial_y^{n-2} q_M(\boldsymbol{x}, y) = \left[\begin{smallmatrix} \boldsymbol{x} \\ y \end{smallmatrix}\right]^\top Q \left[\begin{smallmatrix} \boldsymbol{x} \\ y \end{smallmatrix}\right]$$

is such that Q is Lorentz. **Compute:**

$$\frac{2 \cdot \partial_y^{n-2} q_M(\boldsymbol{x}, y)}{(n-2)!} = n(n-1) \cdot y^2 + 2(n-1) \cdot \sum_{e \in E} x_e y + 2 \cdot \sum_{\{e,f\} \in \mathcal{I}} x_e x_f.$$

Recall: This is scaled version of an independent set polynomial, so we have that $\sum_{\{e,f\} \in \mathcal{I}} x_e x_f = \sum_{\{e,f\} \in \mathcal{B}} x_e x_f$ is CLC ($\mathcal{B}$ of truncated matroid).

So: $Q = \begin{bmatrix} Q_{\mathcal{B}} & (n-1) \cdot \mathbf{1}_E \\ (n-1) \cdot \mathbf{1}_E^\top & n(n-1) \end{bmatrix}$, where $Q_{\mathcal{B}}$ is Lorentz since it corresponds to the basis generating polynomial of the truncated matroid.

Exercise: The matrix Q is also Lorentz.

Outline

Sampling via random walks

Goal: Given a matroid $M = (E, \mathcal{B})$, sample uniformly from $\mathcal{B}$.

Problem: Number of bases is often exponential in $n = |E|$; e.g. there are m^{m-2} spanning trees of the complete graph on $m \approx \sqrt{n}$ vertices.

One approach: The basis exchange graph gives us a way to "walk" to different bases. Given a membership oracle (tells if a given set is a basis or not), we can:

1. Start at some basis B_0.
2. Remove a random element e from B_0.
3. Add a random element f, given that $(B_0 \setminus \{e\}) \cup \{f\} \in \mathcal{B}$.
4. Call this new basis $B_1 := (B_0 \setminus \{e\}) \cup \{f\}$.

Equivalent: Randomly walking along edges of the basis exchange graph.

As the number of iterations/steps increases, the randomness increases.

Eventually: "Random enough" so that B_k is $\approx$ uniformly random.

How good is a random walk?

Good news: Random walk gives an algorithm for $\approx$ uniform sampling.

Problem: What if the basis exchange graph is similar to a path or cycle?

- Starting at one end of the path/cycle means that it will take $O(|\mathcal{B}|)$ steps to even **see** the other end.
- The number of steps needed is at least $O(|\mathcal{B}|) >$ exponential.

However: If graph is complete, then one step suffices. (But $|E| \approx |\mathcal{B}|$.)

Consider the respective transition matrices:

$$T_{C_n} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad T_{K_n} = \frac{1}{n-1}\begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$$

$$\text{eig}(T_{C_n}) = \left(1, \cos(\tfrac{2\pi}{n}), \cos(\tfrac{4\pi}{n}), \ldots\right), \ \text{eig}(T_{K_n}) = \left(1, -\tfrac{1}{n-1}, \ldots, -\tfrac{1}{n-1}\right).$$

Mixing time of a random walk

Cycle graph: $\lambda_2 \approx 1 - \frac{1}{n^2}$, and **complete graph:** $\lambda_2 = -\frac{1}{n-1}$.

Upshot: Second largest eigenvalue of the transition matrix is a measure of how "bottlenecky" the graph is (see also: Cheeger constant).

Roughly: For nice random walks, we have

$$t_{\text{mix}} \leq O_\epsilon\left([1 - \lambda_2(T)]^{-1}\right),$$

where t_{mix} is the **mixing time** of the random walk = number of steps until random walk is close to uniform. **Want $\lambda_2(T)$ to be small.**

Now: The Hessian matrix of basis generating polynomial of a matroid has small second eigenvalue. **Can we relate this to the second eigenvalue of the transition matrix for the random walk?**

Local random walks

First: Let $M = (E, \mathcal{B})$ be a rank-two matroid. Consider the random walk on E (instead of $\mathcal{B}$) with e, f connected by an edge whenever $\{e, f\} \in \mathcal{B}$.

"Dual" to the basis exchange walk:

- Add random element, then remove random element (reverse order).
- **Anari-Liu-Oveis Gharan-Vinzant '19:** Dual walk and basis exchange walk have the same non-zero eigenvalues.

Transition matrix is precisely Q up to scalar, where $p_M(\boldsymbol{x}) = \boldsymbol{x}^\top Q \boldsymbol{x}$ is the basis generating polynomial. **CLC $\implies$ small second eigenvalue.**

Therefore: We have small mixing time for rank-two matroids.

How do we generalize this? By considering minors (contractions and truncations) of any matroid M, we can look at such "local" walks with respect to any independent set $I \in \mathcal{I}$. $\implies$ **Local-to-global theorem.**

Local-to-global theorem

Given a matroid $M = (E, \mathcal{I})$, fix any $I \in \mathcal{I}$ with $|I| = k$. Define:

- $E_I :=$ all independent sets J such that $I \subset J$ and $|J| = k + 1$.
- $\mathcal{B}_I :=$ all independent sets J such that $I \subset J$ and $|J| = k + 2$.

Equivalent: Contract for all $e \in I$, and then truncate to rank two.

In terms of polynomials: $\left(\prod_{e \in I} \partial_{x_e}\right) p_M \implies$ look at Hessian matrix.

Kaufman-Oppenheim '18, Anari-Liu-Oveis Gharan-Vinzant '19: If the second eigenvalue of the transition matrix of the local walk (previous slide) corresponding to I is small for every $I \in \mathcal{I}$, then the second eigenvalue of the transition matrix of the basis exchange walk is small.

Idea: Can "patch" the local walks together to hit all bases.

Note: Original result [KO '18] is for more general simplicial complexes.

Corollary: Matroid basis generating polynomial is CLC $\implies$ small mixing time for the basis exchange walk.

Basis sampling overview

Main idea: Small second eigenvalue of transition matrix implies small mixing time for the random walk.

Kaufman-Oppenheim '18: "Local" second eigenvalues being small implies the "global" second eigenvalue is small.

Anari-Liu-Oveis Gharan-Vinzant '19: "Local" eigenvalues correspond precisely to eigenvalues of the Hessian of some derivatives applied to the "global" generating polynomial.

CLC property precisely captures this information.

In fact: Non-uniform sampling allowed as long as polynomial is CLC.

CLC implies matroid support. $\implies$ This only works for matroids?

Actually: Second eigenvalue ≤ 0 (Lorentz matrices) is stronger than what is actually needed to use the results of [KO '18].

Open question: Is there a theory of CLC-like polynomials where the second eigenvalue is at most some $\epsilon > 0$ (or $\frac{\epsilon}{n} > 0$, or $\frac{\epsilon}{d} > 0$, etc.)?

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Counting the intersection of two matroids

Fact: The intersection of the bases of two matroids is not itself a matroid. The generating polynomial is not CLC.

E.g.: Perfect matchings of a bipartite graph G on vertices $V_1 \sqcup V_2$ with edges E. Define matroids $M_i = (E, \mathcal{B}_i)$ with $\mathcal{B}_i :=$ choices of edges such that each $v \in V_i$ is incident on exactly one edge.

Therefore: Matroid intersection captures the permanent (#P-hard).

One way to count:

$$\langle p, q \rangle := \sum_S p_S q_S \quad \implies \quad \langle p_{M_1}, p_{M_2} \rangle = \#(\mathcal{B}_1 \cap \mathcal{B}_2).$$

We also have $\langle p, q \rangle = \prod_{i=1}^n (1 + \partial_{x_i} \partial_{z_i}) \Big|_{\boldsymbol{x}=\boldsymbol{z}=0} [p(\boldsymbol{x}) \cdot q(\boldsymbol{z})]$.

Question: Can we bound approximate this inner product? Is there a connection to real stability preservers? Algorithmic implications?