Applications of CLC/Lorentzian Polynomials Exercises

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Definition. A matrix Q is said to be **Lorentz** if it is a Hermitian matrix in the close of the set of all matrices with Lorentz signature $(+,-,-,\dots,-)$.

Definition. A polynomial $p \in \mathbb{R}_+[\boldsymbol{x}]$ is said to be **indecomposable** if there is no way to write $p = f + g$ where $f, g \not\equiv 0$ depend on disjoint sets of variables.

Definition. A d-homogeneous polynomial $p \in \mathbb{R}_+[\boldsymbol{x}]$ is said to be **completely log-concave** if for all $k \in \mathbb{Z}_+$ and all choices of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k \in \mathbb{R}^n_+$, we have that

$$\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} p = \left(\prod_{i=1}^k \sum_{j=1}^n v_{ij} \right) p$$

is log-concave in the positive orthant. We also consider the zero polynomial to be completely log-concave.

Theorem. *A d-homogeneous polynomial $p \in \mathbb{R}_+[\boldsymbol{x}]$ is completely log-concave if and only if:*

1. *For all $\boldsymbol{\mu} \in \mathbb{Z}^n_+$ such that $|\mu| \leq d-2$, the polynomial $\partial^{\boldsymbol{\mu}}_{\boldsymbol{x}} p$ is indecomposable.*
2. *For all $\boldsymbol{\mu} \in \mathbb{Z}^n_+$ such that $|\mu| = d-2$, the polynomial $\partial^{\boldsymbol{\mu}}_{\boldsymbol{x}} p = \boldsymbol{x}^\top Q \boldsymbol{x}$ is such that Q is Lorentz.*

Exercises

1. Given a matroid $M = (E, \mathcal{B})$, define the **basis generating polynomial** via

$$p_M(\boldsymbol{x}) := \sum_{B \in \mathcal{B}} \boldsymbol{x}^B,$$

as was done in the course. The operations of **deletion** and **contraction** on M correspond to $|_{x_i=0}$ (evaluation at 0) and ∂_{x_i} on the corresponding polynomials. Describe the corresponding operations on the underlying matroids, and prove that these operations do in fact transfrom M into another matroid.

2. Given a matroid $M = (E, \mathcal{I})$ with $|E| = n$, define the **independent set generating polynomial** via

$$p_M(\boldsymbol{x}, y) := \sum_{I \in \mathcal{I}} y^{n-|I|} \boldsymbol{x}^I,$$

as was done in the course. Finish the proof that this polynomial is completely log-concave. Specifically,

   (a) Describe the action of ∂_{x_i} and ∂_y in terms of the polynomials and the underlying matroids.

   (b) Determine the relationship between ∂_y and matroid **truncation**, which removes all independent sets of size large than some specified bound. Prove that this operation transforms M into another matroid.

(c) Prove the indecomposability condition for p_M.

(d) Compute the quadratic derivatives of p_M and prove they are Lorentz.

3. Consider a slightly different version of the independent set generating function of the previous exercise:

$$p_M(\boldsymbol{x}, y) := \sum_{I \in \mathcal{I}} \frac{y^{n-|I|}}{(n-|I|)!} \boldsymbol{x}^I.$$

Prove that this polynomial is completely log-concave, which should be easier than the previous exercise. What sort of "Mason's conjecture"-type result does this give? Also, can we replace n by r (the rank of M) in this version of the polynomial?

4. Given convex compact sets $K_1, \ldots, K_n$ in $\mathbb{R}^d$, it turns out that

$$p_{\boldsymbol{K}}(\boldsymbol{x}) := \text{vol}\left(\sum_{i=1}^n x_i K_i\right) = \sum_{i_1,\ldots,i_d=1}^n V(K_{i_1}, \ldots, K_{i_d}) \prod_{k=1}^d x_{i_k}$$

is a d-homogeneous polynomial with non-negative coefficients, where $\sum_{i=1}^n x_i K_i$ denotes scaling (for $x_i > 0$) and Minkowski sum, and $V(K_{i_1}, \ldots, K_{i_d})$ is a symmetric multilinear function of convex compact sets in $\mathbb{R}^d$ called the **mixed volume**. A key property of the mixed volume are the **Alexandrov-Fenchel inequalities**, which say that for any convex compact $K_1, \ldots, K_d \subset \mathbb{R}^d$, we have

$$V(K_1, K_2, K_3, \ldots, K_d) \geq \sqrt{V(K_1, K_1, K_3, \ldots, K_d) \cdot V(K_2, K_2, K_3, \ldots, K_d)}.$$

Use this property to show that $p_{\boldsymbol{K}}(\boldsymbol{x})$ is completely log-concave.

5. Let $M = (E, \mathcal{B})$ be a linear matroid (over $\mathbb{R}$), given explicitly by the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{R}^d$. Consider the polynomial

$$p_M(\boldsymbol{x}) := \det\left(M \cdot \text{diag}(\boldsymbol{x}) \cdot M^\top\right), \quad \text{where} \quad M := \begin{bmatrix} | & | & & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \\ | & | & & | \end{bmatrix}$$

and $\text{diag}(\boldsymbol{x})$ is the diagonal matrix with $\boldsymbol{x}$ along the diagonal.

(a) Prove that $p_M(\boldsymbol{x})$ is a real stable multiaffine d-homogeneous polynomial with non-negative coefficients whose support is $\mathcal{B}$, the set of bases of the matroid M.

(b) Does this mean that the basis generating polynomial of every linear matroid over $\mathbb{R}$ is real stable?

(c) Recall the way we proved that the spanning tree generating polynomial of a connected graph is real stable by way of the Laplacian matrix. Prove that all graphic matroids are linear matroids.