Linear Operator Lower Bounds via Capacity

Polynomial Capacity: Theory, Applications, Generalizations

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Notation

Polynomial notation:

- $\mathbb{R}, \mathbb{R}_+, \mathbb{Z}_+ :=$ reals, non-negative reals, non-negative integers.
- $\boldsymbol{x}^{\boldsymbol{\mu}} := \prod_i x_i^{\mu_i}$ and $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ is entrywise.
- $\mathbb{R}[\boldsymbol{x}] :=$ v.s. of real polynomials in n variables.
- $\mathbb{R}_+[\boldsymbol{x}] :=$ v.s. of real polynomials with non-negative coefficients.
- $\mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}] :=$ v.s. of polynomials of degree at most λ_i in x_i.
- For $p \in \mathbb{R}[\boldsymbol{x}]$, we write $p(\boldsymbol{x}) = \sum_{\boldsymbol{\mu}} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$.
- For d-homogeneous $p \in \mathbb{R}[\boldsymbol{x}]$, we write $p(\boldsymbol{x}) = \sum_{|\boldsymbol{\mu}|=d} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$.
- $\frac{d}{dx} = \frac{\partial}{\partial x} = \partial_x :=$ derivative with respect to x, and $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} := \prod_i \partial_{x_i}^{\mu_i}$.
- $\text{supp}(p) =$ **support** of $p =$ the set of $\boldsymbol{\mu} \in \mathbb{Z}_+^n$ for which $p_{\boldsymbol{\mu}} \neq 0$.
- $\text{Newt}(p) =$ **Newton polytope** of $p =$ convex hull of the support of p as a subset of $\mathbb{R}^n$.

Outline

1. Motivation
   - Capacity preserving linear operators
   - Hopeful applications
   - A unified approach
2. Bilinear form bounds
   - A special bilinear form for real stable polynomials
   - Strong Rayleigh inequalities
   - The bound for multiaffine polynomials
   - The bound in full generality
3. Capacity preserving linear operators

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Capacity preserving linear operators

Recall: $\mathrm{Cap}_{\boldsymbol{\alpha}}(p) := \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\boldsymbol{\alpha}}} = \inf_{x_1,\ldots,x_n>0} \frac{p(x_1,\ldots,x_n)}{x_1^{\alpha_1}\cdots x_n^{\alpha_n}}$.

Recall: For real stable $p \in \mathbb{R}_+^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ and $\boldsymbol{\mu} \in \mathbb{Z}_+^n$, we have

$$\frac{1}{\mu_n!} \mathrm{Cap}_{(\mu_1,\ldots,\mu_{n-1})}\left(\partial_{x_n}|_{x_n=0}\, p\right) \geq \binom{\lambda_n}{\mu_n} \frac{\mu_n^{\mu_n}(\lambda_n-\mu_n)^{\lambda_n-\mu_n}}{\lambda_n^{\lambda_n}} \mathrm{Cap}_{\boldsymbol{\mu}}(p).$$

Corollary: For real stable $p \in \mathbb{R}_+^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ and $\boldsymbol{\mu} \in \mathbb{Z}_+^n$, we have

$$p_{\boldsymbol{\mu}} \geq \left[\prod_{i=1}^n \binom{\lambda_i}{\mu_i} \frac{\mu_i^{\mu_i}(\lambda_i-\mu_i)^{\lambda_i-\mu_i}}{\lambda_i^{\lambda_i}}\right] \mathrm{Cap}_{\boldsymbol{\mu}}(p).$$

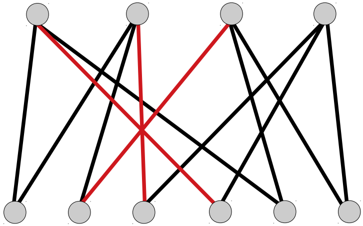
Another interpretation: The first bound is a statement about how much the operator $\partial_{x_n}|_{x_n=0}$ can decrease the capacity of a polynomial.

$\implies$ "$\partial_{x_n}|_{x_n=0}$ **preserves capacity** up to factor $\binom{\lambda_n}{\mu_n}\frac{\mu_n^{\mu_n}(\lambda_n-\mu_n)^{\lambda_n-\mu_n}}{\lambda_n^{\lambda_n}}$".

What other linear operators preserve capacity?

Hopeful application: Non-perfect matchings

Let G be a (a, b)-biregular (m, n)-bipartite graph ($am = bn$) and consider:



Bipartite adjacency matrix, A:

$$\begin{bmatrix} 1 & 0 & 0 & \color{red}{1} & 1 & 0 \\ 1 & 1 & \color{red}{1} & 0 & 0 & 0 \\ 0 & \color{red}{1} & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$\#$ k-matchings $=$ k-subpermanents

(a, b)-regular $\iff$ (a, b)-stochastic

We can associate a polynomial to G in the same way as before:

$$p_G(\mathbf{x}) := \prod_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_j$$

Question: What operator picks out k-matchings?

Hopeful application: Non-perfect matchings

For (a, b)-regular (m, n)-bipartite G with adjacency matrix A:

$$p_G(\boldsymbol{x}) := \prod_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_j$$

$\#$ perfect matchings $=$ permanent $= \partial_{\boldsymbol{x}}^{[n]}\Big|_{\boldsymbol{x}=\boldsymbol{0}} p_G$

$\#$ k-matchings $=$ sum of k-subpermanents $\approx \sum_{S \in \binom{[n]}{k}} \partial_{\boldsymbol{x}}^{S}\Big|_{\boldsymbol{x}=\boldsymbol{1}} p_G$

Evaluation at 1 instead of 0: requires regularity of the graph.

$$\sum_{S \in \binom{[n]}{k}} \partial_{\boldsymbol{x}}^{S}\Big|_{\boldsymbol{x}=\boldsymbol{1}} p_G = \mu_k(G) \cdot a^{m-k} \qquad (a \text{ is the row sum})$$

Why? $\sum_S \partial_{\boldsymbol{x}}^S p_G$ is $(m-k)$-homogeneous, so evaluation at $\boldsymbol{0}$ is no good.

Problem: The operator $\sum_{S \in \binom{[n]}{k}} \partial_{\boldsymbol{x}}^{S}\Big|_{\boldsymbol{x}=\boldsymbol{1}}$ does not pick out a coefficient, and there is no clear way to induct like with the coefficient bounds.

Hopeful application: Intersection of two matroids

Recall: A **matroid** M on a ground set E can be defined as a non-empty collection $\mathcal{B}$ of size-d subsets of E called **bases** of M, which satisfy:

- **Exchange axiom:** For all $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$, there exists $f \in B_2 \setminus B_1$ such that $B_1 \cup \{f\} \setminus \{e\} \in \mathcal{B}$.

E.g.: Linear bases in a collection of vectors, spanning trees of a graph.

Recall: For any matroid, the basis-generating polynomial is Lorentzian:

$$p_M(\boldsymbol{x}) := \sum_{B \in \mathcal{B}} \boldsymbol{x}^B = \sum_{B \in \mathcal{B}} \prod_{e \in B} x_e \in \mathbb{R}_+^{\mathbf{1}}[\boldsymbol{x}].$$

Matroid intersection problem: Given matroids M_1, M_2 on the same ground set E, count or approximate $|\mathcal{B}_1 \cap \mathcal{B}_2|$.

Problem: Not clear how to induct, and not clear how this could be interpreted as a coefficient of something.

Hint: Something like a sum of coefficients (the matchings case too).

A unified approach

The two previous examples are both **weighted sums of coefficients**.

How to unify? First idea: **Inner product** of polynomials:

$$\langle p, q \rangle = \sum_{\mu} c_{\mu} p_{\mu} q_{\mu}$$

- Works well for the matroid intersection case: just plug in the matroid-generating polynomials with $c_{\mu} = 1$.
- Non-perfect matchings case is not as clear, but perhaps plugging in something like $\sum_{S \in \binom{[n]}{k}} \boldsymbol{x}^S$ should work with c_{μ} some factorials? This is the elementary symmetric polynomial, which is real stable.

There is a **bilinear form** we discussed at the beginning of the course which plays very nicely with log-concave polynomials.

Problem: Not quite an inner product. But maybe it's close enough?

Outline

A bilinear form for real stable polynomials

Given real stable polynomials $p, q \in \mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$, define:

$$\langle p, q \rangle_{\boldsymbol{\lambda}} := \sum_{\mathbf{0} \leq \boldsymbol{\mu} \leq \boldsymbol{\lambda}} \binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}}^{-1} p_{\boldsymbol{\mu}} q_{\boldsymbol{\lambda} - \boldsymbol{\mu}}.$$

Note: Use of $\langle \cdot, \cdot \rangle_{\boldsymbol{\lambda}}$ should give the intuition of an inner product, but really it is something like a "twisted" inner product.

Why this? Recall: This bilinear form is associated to the Borcea-Brändén symbol of a linear operator. (We will come back to this.)

For **multiaffine** $p, q \in \mathbb{R}^{\mathbf{1}}[\boldsymbol{x}]$, we have:

$$\langle p, q \rangle_{\mathbf{1}} := \sum_{\mathbf{0} \leq \boldsymbol{\mu} \leq \mathbf{1}} p_{\boldsymbol{\mu}} q_{\mathbf{1} - \boldsymbol{\mu}} = \left[\prod_{i=1}^{n} (\partial_{x_i} + \partial_{z_i}) \Big|_{x_i = z_i = 0} \right] p(\boldsymbol{x}) q(\boldsymbol{z}).$$

Goal: Lower bound on this bilinear form in terms of the capacity of p, q.

Idea for multiaffine: Product $p(\boldsymbol{x}) q(\boldsymbol{z})$ is real stable, $(\partial_{x_i} + \partial_{z_i})\big|_{x_i = z_i = 0}$ preserves real stability, proof goes by induction. (Standard stuff.)

The base case for multiaffine

For **multiaffine** $p, q \in \mathbb{R}_+^{\mathbf{1}}[\boldsymbol{x}]$, we have:

$$\langle p, q \rangle_{\mathbf{1}} := \sum_{\mathbf{0} \leq \boldsymbol{\mu} \leq \mathbf{1}} p_{\boldsymbol{\mu}} q_{\mathbf{1}-\boldsymbol{\mu}} = \left[\prod_{i=1}^{n} (\partial_{x_i} + \partial_{z_i}) \Big|_{x_i = z_i = 0} \right] p(\boldsymbol{x}) q(\boldsymbol{z}).$$

What does the base case look like?

$$\left[\prod_{i=1}^{n-1} (\partial_{x_i} + \partial_{z_i}) \Big|_{x_i = z_i = 0} \right] p(\boldsymbol{x}) q(\boldsymbol{z}) = a x_n z_n + b x_n + c z_n + d \in \mathbb{R}_+^{(1,1)}[x_n, z_n].$$

Base case: Given $p(x, z) = axz + bx + cz + d \in \mathbb{R}_+^{(1,1)}[x, z]$ and $\alpha \in [0, 1]$, we want a bound like

$$b + c = (\partial_x + \partial_z) \Big|_{x=z=0} p \geq K(\alpha) \cdot \text{Cap}_{(\alpha, 1-\alpha)}(p).$$

Questions: How does log-concavity (or something related to real stability) come into play? Why α and $1 - \alpha$ here?

The strong Rayleigh inequalities

Strong Rayleigh inequalities [Brändén '07]: For real stable $p \in \mathbb{R}^{\mathbf{1}}[\boldsymbol{x}]$,

$$\partial_{x_i} p(\boldsymbol{x}) \cdot \partial_{x_j} p(\boldsymbol{x}) - p(\boldsymbol{x}) \cdot \partial_{x_i} \partial_{x_j} p(\boldsymbol{x}) \geq 0 \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^d.$$

Converse: This condition for all i, j is equivalent to real stability in $\mathbb{R}^{\mathbf{1}}[\boldsymbol{x}]$.

What about our base case? For $p(x,z) = axz + bx + cz + d$, we have

$$\partial_x p \cdot \partial_z p - p \cdot \partial_x \partial_z p = (az+b)(ax+c) - a(axz+bx+cz+d) = bc - ad.$$

Corollary: $axz + bx + cz + d$ is real stable iff $bc \geq ad$.

This is analogous to the discriminant condition for univariate quadratics.

In fact: Polarize $f(x) = ax^2 + 2bx + c$ to get $p(x,z) = axz + bx + bz + c$, for which: p is real stable iff $b^2 \geq ac$ iff f is real-rooted.

Recall: Polarization is the unique mutliaffine symmetric polynomial p in $d = 2$ variables which diagonalizes to f.

Strong Rayleigh and capacity: The separation trick

Previous slide: $axz + bx + cz + d$ is real stable iff $bc \geq ad$.

How to use this for capacity? We use the following **separation trick**:

$$\inf_{x,z>0} \frac{axz + bx + cz + d}{x^\alpha z^{1-\alpha}} \leq \inf_{x,z>0} \frac{axz + bx + cz + \frac{bc}{a}}{x^\alpha z^{1-\alpha}}$$
$$= \inf_{x,z>0} \frac{(az + b)(x + \frac{c}{a})}{x^\alpha z^{1-\alpha}}$$

Separate via $\inf_{x,z>0} \frac{(az+b)(x+\frac{c}{a})}{x^\alpha z^{1-\alpha}} = \mathrm{Cap}_{1-\alpha}(az + b) \cdot \mathrm{Cap}_\alpha(x + \frac{c}{a})$.

Calculus: For any $r, s > 0$, we have $\mathrm{Cap}_\alpha(rx + s) = \frac{r^\alpha s^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}}$. **So:**

$$\mathrm{Cap}_{1-\alpha}(az + b) \cdot \mathrm{Cap}_\alpha(x + \tfrac{c}{a}) = \frac{a^{1-\alpha} b^\alpha}{(1-\alpha)^{1-\alpha}\alpha^\alpha} \cdot \frac{1^\alpha (\frac{c}{a})^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}}$$
$$= \frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}} \cdot \mathrm{Cap}_\alpha(bx + c).$$

Finally: $\mathrm{Cap}_\alpha(bx + c) = \inf_{x>0} \frac{bx+c}{x^\alpha} \leq \frac{b \cdot 1 + c}{1^\alpha} = b + c$.

Putting it all together (the base case)

Last slide: For real stable $p(x,z) = axz + bx + cz + d$ and $\alpha \in [0,1]$,

1. $\mathrm{Cap}_{(\alpha,1-\alpha)}(p) \leq \mathrm{Cap}_{1-\alpha}(az+b) \cdot \mathrm{Cap}_{\alpha}(x + \frac{c}{a})$
2. $\mathrm{Cap}_{1-\alpha}(az+b) \cdot \mathrm{Cap}_{\alpha}(x + \frac{c}{a}) = \frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \cdot \mathrm{Cap}_{\alpha}(bx+c)$
3. $\mathrm{Cap}_{\alpha}(bx+c) \leq b + c$

Combine: $(\partial_x + \partial_z)p\Big|_{x=z=0} = b + c \geq \alpha^{\alpha}(1-\alpha)^{1-\alpha} \cdot \mathrm{Cap}_{(\alpha,1-\alpha)}(p)$.

Lemma (Base case, Anari-Oveis Gharan '17)

Given a real stable polynomial $p \in \mathbb{R}_+^{(1,1)}[x,z]$ and any $\alpha \in [0,1]$, we have

$$(\partial_x + \partial_z)p\Big|_{x=z=0} \geq \alpha^{\alpha}(1-\alpha)^{1-\alpha} \cdot \mathrm{Cap}_{(\alpha,1-\alpha)}(p).$$

Additionally, this bound is tight for any fixed $\alpha \in [0,1]$.

The bound for multiaffine polynomials in general

Lemma: $(\partial_x + \partial_z)p\Big|_{x=z=0} \geq \alpha^\alpha (1-\alpha)^{1-\alpha} \cdot \mathrm{Cap}_{(\alpha,1-\alpha)}(p)$.

Theorem (Mutliaffine bound, Anari-Oveis Gharan '17)

Given real stable polynomials $p, q \in \mathbb{R}_+^{\mathbf{1}}[\boldsymbol{x}]$ and any $\boldsymbol{\alpha} \in [0,1]^n$, we have

$$\langle p, q\rangle_{\mathbf{1}} \geq \left[\prod_{i=1}^n \alpha_i^{\alpha_i}(1-\alpha_i)^{1-\alpha_i}\right] \cdot \mathrm{Cap}_{\boldsymbol{\alpha}}(p) \cdot \mathrm{Cap}_{\mathbf{1}-\boldsymbol{\alpha}}(q).$$

Additionally, this bound is tight for any fixed $\boldsymbol{\alpha} \in [0,1]^n$.

Proof strategy: Induction with partial evaluation, per usual.

Want: For real stable $f(\boldsymbol{x},\boldsymbol{z}) \in \mathbb{R}_+^{(\mathbf{1},\mathbf{1})}[\boldsymbol{x},\boldsymbol{z}]$ (think $f(\boldsymbol{x},\boldsymbol{z}) = p(\boldsymbol{x})q(\boldsymbol{z})$),

$$\left[\prod_{i=1}^n (\partial_{x_i} + \partial_{z_i})\Big|_{x_i=z_i=0}\right] f(\boldsymbol{x},\boldsymbol{z}) \geq \left[\prod_{i=1}^n \alpha_i^{\alpha_i}(1-\alpha_i)^{1-\alpha_i}\right] \cdot \mathrm{Cap}_{(\boldsymbol{\alpha},\mathbf{1}-\boldsymbol{\alpha})}(f).$$

Proof of the multiaffine bound

To prove: For real stable $f(\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}_+^{(\mathbf{1},\mathbf{1})}[x_1, \ldots, x_n, z_1, \ldots, z_n]$,

$$\left[\prod_{i=1}^{n} (\partial_{x_i} + \partial_{z_i})\Big|_{x_i=z_i=0}\right] f(\boldsymbol{x}, \boldsymbol{z}) \geq \left[\prod_{i=1}^{n} \alpha_i^{\alpha_i}(1-\alpha_i)^{1-\alpha_i}\right] \cdot \mathrm{Cap}_{(\boldsymbol{\alpha},\mathbf{1}-\boldsymbol{\alpha})}(f).$$

Define $D_i := (\partial_{x_i} + \partial_{z_i})\Big|_{x_i=z_i=0}$ and $C_i := \alpha_i^{\alpha_i}(1-\alpha_i)^{1-\alpha_i}$.

Induction on n: For $\beta := (\alpha_1, \ldots, \alpha_{n-1})$, apply bound to $D_n f$:

$$\left(\prod_{i=1}^{n-1} D_i\right) D_n f \geq \left(\prod_{i=1}^{n-1} C_i\right) \cdot \mathrm{Cap}_{(\boldsymbol{\beta},\mathbf{1}-\boldsymbol{\beta})}(D_n f).$$

Next: $\left[(\prod_{i=1}^{n-1} D_i) f\right](x_n, z_n)$ is real stable since D_i preserves stability.

Also: For fixed $\boldsymbol{x}' := (x_1, \ldots, x_{n-1}) > 0$ and $\boldsymbol{z}' := (z_1, \ldots, z_{n-1}) > 0$, we have that $f(\boldsymbol{x}', x_n, \boldsymbol{z}', z_n) \in \mathbb{R}_+[x_n, z_n]$ is real stable, and base case gives

$$D_n \left[f(\boldsymbol{x}', x_n, \boldsymbol{z}', z_n)\right] \geq C_n \cdot \mathrm{Cap}_{(\alpha_n, 1-\alpha_n)} \left(f(\boldsymbol{x}', x_n, \boldsymbol{z}', z_n)\right).$$

Putting it all together (the general multiaffine case)

Given real stable $f(\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}_+^{(\mathbf{1},\mathbf{1})}[\boldsymbol{x}, \boldsymbol{z}]$ and $\boldsymbol{\alpha} \in [0,1]^n$, we have

1. **Induction:** $\left(\prod_{i=1}^{n-1} D_i\right) D_n f \geq \left(\prod_{i=1}^{n-1} C_i\right) \cdot \mathrm{Cap}_{(\beta,\mathbf{1}-\beta)}(D_n f)$
2. **Final step:** $D_n f(\boldsymbol{x}', \boldsymbol{z}') \geq C_n \cdot \mathrm{Cap}_{(\alpha_n, 1-\alpha_n)}\left(f(\boldsymbol{x}', x_n, \boldsymbol{z}', z_n)\right)$

Now combine (recall $\beta = (\alpha_1, \ldots, \alpha_{n-1})$):

$$\left(\prod_{i=1}^{n} D_i\right) f \geq \left(\prod_{i=1}^{n-1} C_i\right) \cdot \inf_{\boldsymbol{x}',\boldsymbol{z}'>0} \frac{D_n f(\boldsymbol{x}', \boldsymbol{z}')}{(\boldsymbol{x}')^{\beta}(\boldsymbol{z}')^{\mathbf{1}-\beta}}$$

$$\geq \left(\prod_{i=1}^{n} C_i\right) \cdot \inf_{\boldsymbol{x}',\boldsymbol{z}'>0} \frac{\inf_{x_n,z_n>0} \frac{f(\boldsymbol{x}', x_n, \boldsymbol{z}', z_n)}{x_n^{\alpha_n} z_n^{1-\alpha_n}}}{(\boldsymbol{x}')^{\beta}(\boldsymbol{z}')^{\mathbf{1}-\beta}}$$

$$= \left(\prod_{i=1}^{n} C_i\right) \cdot \inf_{\boldsymbol{x},\boldsymbol{z}>0} \frac{f(\boldsymbol{x}, \boldsymbol{z})}{\boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{z}^{\mathbf{1}-\boldsymbol{\alpha}}} = \left(\prod_{i=1}^{n} C_i\right) \cdot \mathrm{Cap}_{(\boldsymbol{\alpha},\mathbf{1}-\boldsymbol{\alpha})}(f).$$

Thus: $f = p(\boldsymbol{x})q(\boldsymbol{z})$ gives $\langle p, q \rangle_{\mathbf{1}} \geq \boldsymbol{\alpha}^{\boldsymbol{\alpha}}(\mathbf{1}-\boldsymbol{\alpha})^{\mathbf{1}-\boldsymbol{\alpha}} \cdot \mathrm{Cap}_{\boldsymbol{\alpha}}(p) \cdot \mathrm{Cap}_{\mathbf{1}-\boldsymbol{\alpha}}(q)$.

The bound in full generality

Theorem: $\langle p, q \rangle_{\mathbf{1}} \geq \boldsymbol{\alpha}^{\boldsymbol{\alpha}}(\mathbf{1} - \boldsymbol{\alpha})^{\mathbf{1}-\boldsymbol{\alpha}} \cdot \mathrm{Cap}_{\boldsymbol{\alpha}}(p) \cdot \mathrm{Cap}_{\mathbf{1}-\boldsymbol{\alpha}}(q)$.

Theorem (Gurvits-L '18)

Given real stable polynomials $p, q \in \mathbb{R}_+^{\boldsymbol{\lambda}}[\mathbf{x}]$ and any $\boldsymbol{\alpha} \in \mathbb{R}_+^n$, we have

$$\langle p, q \rangle_{\boldsymbol{\lambda}} \geq \left[\prod_{i=1}^n \frac{\alpha_i^{\alpha_i}(\lambda_i - \alpha_i)^{\lambda_i - \alpha_i}}{\lambda_i^{\lambda_i}}\right] \cdot \mathrm{Cap}_{\boldsymbol{\alpha}}(p) \cdot \mathrm{Cap}_{\boldsymbol{\lambda}-\boldsymbol{\alpha}}(q).$$

Additionally, this bound is tight for any fixed $\boldsymbol{\alpha} \in \mathbb{R}_+^n$.

Proof strategy: Polarization preserves real stability **and capacity**.

Degree-agnostic version: For real stable $p, q \in \mathbb{R}_+[\mathbf{x}]$ and any $\boldsymbol{\alpha} \in \mathbb{R}_+^n$,

$$\langle p, q \rangle_\infty := p(\partial_{\mathbf{x}})q(\mathbf{x})\Big|_{\mathbf{x}=\mathbf{0}} \geq \boldsymbol{\alpha}^{\boldsymbol{\alpha}} e^{-\boldsymbol{\alpha}} \cdot \mathrm{Cap}_{\boldsymbol{\alpha}}(p) \cdot \mathrm{Cap}_{\boldsymbol{\alpha}}(q).$$

Note: The inner product $\langle p, q \rangle_\infty$ is an actual inner product. The above theorem can also be "untwisted" by mapping $q \mapsto \mathbf{x}^{\boldsymbol{\lambda}} \cdot q(x_1^{-1}, \ldots, x_n^{-1})$.

Outline

From the bilinear form to linear operators

Recall: The symbol of an operator $T : \mathbb{R}^{\boldsymbol{\lambda}}[[\boldsymbol{x}] \to \mathbb{R}[\boldsymbol{x}]$:

$$\mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x}, \boldsymbol{y}) := T\left[\prod_{i=1}^{n}(x_i + y_i)^{\lambda_i}\right] = \sum_{\boldsymbol{0} \leq \boldsymbol{\mu} \leq \boldsymbol{\lambda}} \binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}} \boldsymbol{y}^{\boldsymbol{\lambda} - \boldsymbol{\mu}} T[\boldsymbol{x}^{\boldsymbol{\mu}}].$$

Here T acts only on the $\boldsymbol{x}$ variables.

Recall: Morally, T preserves real stability iff $\mathrm{Symb}^{\boldsymbol{\lambda}}[T]$ is real stable.

How was this proven? We used $\langle \cdot, \cdot \rangle_{\boldsymbol{\lambda}}$ via the fact that

$$T[p](\boldsymbol{x}) = \left\langle \mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x}, \boldsymbol{y}),\ p(\boldsymbol{y}) \right\rangle_{\boldsymbol{\lambda}}$$

where $\langle \cdot, \cdot \rangle_{\boldsymbol{\lambda}}$ acts on the $\boldsymbol{y}$ variables.

Question: Can we use this to prove capacity bounds for linear operators?

In particular, this will help with the non-perfect matchings application.

Capacity bounds for linear operators

Last slide:

$$T[p](\boldsymbol{x}) = \left\langle \mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x},\boldsymbol{y}),\ p(\boldsymbol{y}) \right\rangle_{\boldsymbol{\lambda}}.$$

If $\mathrm{Symb}^{\boldsymbol{\lambda}}[T]$ is real stable, then for any $\boldsymbol{\alpha} > 0$ and any fixed $\boldsymbol{x} > 0$ we have

$$\begin{aligned} T[p](\boldsymbol{x}) &= \left\langle \mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x},\boldsymbol{y}),\ p(\boldsymbol{y}) \right\rangle_{\boldsymbol{\lambda}} \\ &\geq \frac{\boldsymbol{\alpha}^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}-\boldsymbol{\alpha})^{\boldsymbol{\lambda}-\boldsymbol{\alpha}}}{\boldsymbol{\lambda}^{\boldsymbol{\lambda}}} \cdot \mathrm{Cap}_{\boldsymbol{\alpha}}(p) \cdot \mathrm{Cap}_{\boldsymbol{\lambda}-\boldsymbol{\alpha}}(\mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x},\cdot)). \end{aligned}$$

For any $\boldsymbol{\beta} > 0$, divide by $\boldsymbol{x}^{\boldsymbol{\beta}}$ and take inf:

$$\inf_{\boldsymbol{x}>0} \frac{T[p](\boldsymbol{x})}{\boldsymbol{x}^{\boldsymbol{\beta}}} \geq \frac{\boldsymbol{\alpha}^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}-\boldsymbol{\alpha})^{\boldsymbol{\lambda}-\boldsymbol{\alpha}}}{\boldsymbol{\lambda}^{\boldsymbol{\lambda}}} \cdot \mathrm{Cap}_{\boldsymbol{\alpha}}(p) \cdot \inf_{\boldsymbol{x}>0} \frac{\mathrm{Cap}_{\boldsymbol{\lambda}-\boldsymbol{\alpha}}(\mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x},\cdot))}{\boldsymbol{x}^{\boldsymbol{\beta}}}.$$

Theorem [Gurvits-L '18]:
If p and $\mathrm{Symb}^{\boldsymbol{\lambda}}[T]$ are real stable, then for any $\boldsymbol{\alpha},\boldsymbol{\beta} > 0$ we have

$$\frac{\mathrm{Cap}_{\boldsymbol{\beta}}(T[p])}{\mathrm{Cap}_{\boldsymbol{\alpha}}(p)} \geq \frac{\boldsymbol{\alpha}^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}-\boldsymbol{\alpha})^{\boldsymbol{\lambda}-\boldsymbol{\alpha}}}{\boldsymbol{\lambda}^{\boldsymbol{\lambda}}} \mathrm{Cap}_{(\boldsymbol{\beta},\boldsymbol{\lambda}-\boldsymbol{\alpha})}(\mathrm{Symb}^{\boldsymbol{\lambda}}[T]).$$