# Linear Operator Bounds Exercises

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**Note:** Exercises with a $*$ next to them are more challenging.

**Definition.** Given a polynomial $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ and $\boldsymbol{\alpha} \in \mathbb{R}_+^n$, we define

$$\mathrm{Cap}_{\boldsymbol{\alpha}}(p) := \inf_{\boldsymbol{x} > 0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\boldsymbol{\alpha}}}.$$

**Definition.** Given two polynomials $p, q \in \mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$, we define

$$\langle p, q \rangle_{\boldsymbol{\lambda}} := \sum_{\mathbf{0} \leq \boldsymbol{\mu} \leq \boldsymbol{\lambda}} \left[ \prod_{i=1}^n \binom{\lambda_i}{\mu_i} \right]^{-1} p_{\boldsymbol{\mu}} q_{\boldsymbol{\mu}}.$$

**Theorem** (Bilinear form capacity bounds). *Given real stable polynomials $p, q \in \mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ and any $\boldsymbol{\alpha}$ such that $\mathbf{0} \leq \boldsymbol{\alpha} \leq \boldsymbol{\lambda}$, we have*

$$\langle p, q \rangle_{\boldsymbol{\lambda}} \geq \left[ \prod_{i=1}^n \frac{\alpha_i^{\alpha_i} (\lambda_i - \alpha_i)^{\lambda_i - \alpha_i}}{\lambda_i^{\lambda_i}} \right] \cdot \mathrm{Cap}_{\boldsymbol{\alpha}}(p) \cdot \mathrm{Cap}_{\boldsymbol{\lambda} - \boldsymbol{\alpha}}(q).$$

**Theorem** (Linear operator capacity bounds). *Given a linear operator $T : \mathbb{R}_+^{\boldsymbol{\lambda}}[\boldsymbol{x}] \to \mathbb{R}_+[\boldsymbol{x}]$ with real stable symbol, a real stable polynomial $P \in \mathbb{R}_+^{\boldsymbol{\lambda}}[\boldsymbol{x}]$, and any $\boldsymbol{\alpha}, \boldsymbol{\beta} > 0$, we have*

$$\frac{\mathrm{Cap}_{\boldsymbol{\beta}}(T[p])}{\mathrm{Cap}_{\boldsymbol{\alpha}}(p)} \geq \left[ \prod_{i=1}^n \frac{\alpha_i^{\alpha_i} (\lambda_i - \alpha_i)^{\lambda_i - \alpha_i}}{\lambda_i^{\lambda_i}} \right] \cdot \mathrm{Cap}_{(\boldsymbol{\beta}, \boldsymbol{\lambda} - \boldsymbol{\alpha})}(\mathrm{Symb}^{\boldsymbol{\lambda}}[T]),$$

*where*

$$\mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x}, \boldsymbol{z}) := T\left[ \prod_{i=1}^n (x_i + z_i)^{\lambda_i} \right] = \sum_{\mathbf{0} \leq \boldsymbol{\mu} \leq \boldsymbol{\lambda}} \left[ \prod_{i=1}^n \binom{\lambda_i}{\mu_i} \right] \boldsymbol{z}^{\boldsymbol{\lambda} - \boldsymbol{\mu}} T[\boldsymbol{x}^{\boldsymbol{\mu}}].$$

## Exercises

1. Use either of the above theorems to reprove the coefficient capacity bounds for real stable polynomials.

2. Prove the general bilinear form capacity bound in the univariate case from the multiaffine bound ($\boldsymbol{\lambda} = \mathbf{1}$). That is, use the bilinear form bounds for $\boldsymbol{\lambda} = \mathbf{1}$ to prove that for univariate $p, q \in \mathbb{R}^d[t]$ and $\alpha \in [0, d]$, we have

   $$\langle p, q \rangle_d \geq \frac{\alpha^\alpha (d - \alpha)^{d - \alpha}}{d^d} \cdot \mathrm{Cap}_\alpha(p) \cdot \mathrm{Cap}_{d - \alpha}(q).$$

   This requires three steps.

   (a) Let $\mathrm{Pol}^d[p]$ be defined as the unique symmetric polynomial $P \in \mathbb{R}^{\mathbf{1}}[x_1, \ldots, x_d]$ such that $P(t, t, \ldots, t) = p(t)$. Prove that

   $$\langle p, q \rangle_d = \langle \mathrm{Pol}^d(p), \mathrm{Pol}^d(q) \rangle_{\mathbf{1}}.$$

(b) For any symmetric polynomial $f \in \mathbb{R}_+[x_1, \ldots, x_n]$ and any $\alpha > 0$, prove that

$$\mathrm{Cap}_{\alpha \cdot \mathbf{1}}(f) = \mathrm{Cap}_{n \cdot \alpha}\big(f(x, x, \ldots, x)\big).$$

(c) Prove the general univariate bound from the multiaffine bound ($\boldsymbol{\lambda} = \mathbf{1}$).

3. Prove the following basic capacity result that has been used throughout the course so far. Given any $\boldsymbol{c} \in \mathbb{R}_+^n$ and $\boldsymbol{\alpha} \in \mathbb{R}_+^n$, we have

$$\mathrm{Cap}_{\boldsymbol{\alpha}}\left[\left(\sum_{i=1}^n c_i x_i\right)^d\right] = \prod_{i=1}^n \left(\frac{d \cdot c_i}{\alpha_i}\right)^{\alpha_i}$$

as long as $\sum_{i=1}^n \alpha_i = d$. What is the capacity value when $\sum_{i=1}^n \alpha_i \neq d$?

4. Prove a capacity bound for the partial derivative $\partial_{x_i}$, without the setting $x_i = 0$ as in Gurvits' theorem.

5. Prove a capacity bound for the operator $\mathrm{MAP} : \mathbb{R}_+^{\boldsymbol{\lambda}}[\boldsymbol{x}] \to \mathbb{R}_+^{\mathbf{1}}[\boldsymbol{x}]$, which simply keeps only the terms which are of degree at most one in each variable and zeros out the rest. Prove also that this map has real stable symbol, so that the above results can even be used.

6. Recall that we used the map MAP to construct the multivariate matching polynomial of a general (not necessarily bipartite) graph via

$$M_G(\boldsymbol{x}) = \mathrm{MAP}\left[\prod_{(u,v) \in E} (1 - x_u x_v)\right].$$

Why can't we use this and the previous exercise to give capacity bounds for matchings of general graphs?

7. ** Is there any way to remedy the situation in the previous problem, in order to be able to use capacity techniques on the general multivariate matching polynomial?