Linear Operator Capacity Bounds Applications

Polynomial Capacity: Theory, Applications, Generalizations

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Notation

Polynomial notation:

- $\mathbb{R}, \mathbb{R}_+, \mathbb{Z}_+ :=$ reals, non-negative reals, non-negative integers.
- $\boldsymbol{x}^{\boldsymbol{\mu}} := \prod_i x_i^{\mu_i}$ and $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ is entrywise.
- $\mathbb{R}[\boldsymbol{x}] :=$ v.s. of real polynomials in n variables.
- $\mathbb{R}_+[\boldsymbol{x}] :=$ v.s. of real polynomials with non-negative coefficients.
- $\mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}] :=$ v.s. of polynomials of degree at most λ_i in x_i.
- For $p \in \mathbb{R}[\boldsymbol{x}]$, we write $p(\boldsymbol{x}) = \sum_{\boldsymbol{\mu}} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$.
- For d-homogeneous $p \in \mathbb{R}[\boldsymbol{x}]$, we write $p(\boldsymbol{x}) = \sum_{|\boldsymbol{\mu}|=d} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$.
- $\frac{d}{dx} = \frac{\partial}{\partial x} = \partial_x :=$ derivative with respect to x, and $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} := \prod_i \partial_{x_i}^{\mu_i}$.
- $\text{supp}(p) =$ **support** of $p =$ the set of $\boldsymbol{\mu} \in \mathbb{Z}_+^n$ for which $p_{\boldsymbol{\mu}} \neq 0$.
- $\text{Newt}(p) =$ **Newton polytope** of $p =$ convex hull of the support of p as a subset of $\mathbb{R}^n$.

Outline

1. Last Time
   - "Inner product" and linear operator capacity bounds
   - Sanity check: Gurvits' theorem
2. Application: Non-perfect matchings
   - Non-perfect matchings of biregular graphs
   - Stability and capacity of the counting operator
   - The bound on non-perfect matchings
3. Application: Intersection of two matroids
   - Matroids and basis intersection
   - A probabilistic approach
   - Using the inner product bound
   - The algorithm
4. Open questions

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"Inner product" capacity bounds

Recall: Given $p \in \mathbb{R}_+[\boldsymbol{x}]$ and $\boldsymbol{\alpha} > 0$, we define

$$\mathrm{Cap}_{\boldsymbol{\alpha}}(p) := \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\boldsymbol{\alpha}}} = \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{x})}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}.$$

Recall: Given $p, q \in \mathbb{R}_+^{\boldsymbol{\lambda}}[\boldsymbol{x}]$, we define

$$\langle p, q \rangle_{\boldsymbol{\lambda}} := \sum_{\boldsymbol{0} \leq \boldsymbol{\mu} \leq \boldsymbol{\lambda}} \binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}} p_{\boldsymbol{\mu}} q_{\boldsymbol{\lambda}-\boldsymbol{\mu}} = \sum_{\boldsymbol{0} \leq \boldsymbol{\mu} \leq \boldsymbol{\lambda}} \binom{\lambda_1}{\mu_1} \cdots \binom{\lambda_n}{\mu_n} p_{\boldsymbol{\mu}} q_{\boldsymbol{\lambda}-\boldsymbol{\mu}}.$$

Theorem (Anari-Oveis Gharan '17, Gurvits-L '18)

Given real stable polynomials $p, q \in \mathbb{R}_+^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ and any $\boldsymbol{\alpha} > 0$, we have

$$\langle p, q \rangle_{\boldsymbol{\lambda}} \geq \left[\prod_{i=1}^{n} \frac{\alpha_i^{\alpha_i} (\lambda_i - \alpha_i)^{\lambda_i - \alpha_i}}{\lambda_i^{\lambda_i}} \right] \mathrm{Cap}_{\boldsymbol{\alpha}}(p) \, \mathrm{Cap}_{\boldsymbol{\lambda}-\boldsymbol{\alpha}}(q).$$

Note: Can "untwist" inner product to get bound with $\mathrm{Cap}_{\boldsymbol{\alpha}}(p) \, \mathrm{Cap}_{\boldsymbol{\alpha}}(q)$.

Linear operator capacity bounds

Recall: Given a linear operator $T : \mathbb{R}_+^{\boldsymbol{\lambda}}[\boldsymbol{x}] \to \mathbb{R}_+[\boldsymbol{x}]$, define

$$\mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x}, \boldsymbol{z}) := T\left[\prod_{i=1}^{n}(x_i + z_i)^{\lambda_i}\right] = \sum_{\boldsymbol{0} \le \boldsymbol{\mu} \le \boldsymbol{\lambda}} \binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}} \boldsymbol{z}^{\boldsymbol{\lambda}-\boldsymbol{\mu}} T[\boldsymbol{x}^{\boldsymbol{\mu}}].$$

Theorem (Gurvits-L '18)

Given real stable $p \in \mathbb{R}_+^{\boldsymbol{\lambda}}[\boldsymbol{x}]$, a linear operator $T : \mathbb{R}_+^{\boldsymbol{\lambda}}[\boldsymbol{x}] \to \mathbb{R}_+[\boldsymbol{x}]$ with real stable symbol, and any sensible $\boldsymbol{\alpha}, \boldsymbol{\beta} > 0$, we have

$$\frac{\mathrm{Cap}_{\boldsymbol{\beta}}(T[p])}{\mathrm{Cap}_{\boldsymbol{\alpha}}(p)} \ge \left[\prod_{i=1}^{n} \frac{\alpha_i^{\alpha_i}(\lambda_i - \alpha_i)^{\lambda_i - \alpha_i}}{\lambda_i^{\lambda_i}}\right] \mathrm{Cap}_{(\boldsymbol{\beta}, \boldsymbol{\lambda}-\boldsymbol{\alpha})}(\mathrm{Symb}^{\boldsymbol{\lambda}}[T]).$$

Further, this bound is tight for given $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\lambda}, T$.

Sanity check: What happens if $T = \partial_{x_n}|_{x_n=0}$ (Gurvits' thoerem)?

Recall: T acts on n-homogeneous polynomials in n variables. Best general choice is $\boldsymbol{\lambda} = (n, n, \ldots, n)$. **Note:** We could choose better $\boldsymbol{\lambda}$ if possible.

Sanity check: Gurvits' theorem

If $T = \partial_{x_n}|_{x_n=0}$ and $\boldsymbol{\lambda} = n \cdot \mathbf{1}$, then

$$\mathrm{Symb}^{n\cdot\mathbf{1}}[T](\boldsymbol{x},\boldsymbol{z}) = T\left[\prod_{i=1}^{n}(x_i+z_i)^n\right] = n \cdot z_n^{n-1}\prod_{i=1}^{n-1}(x_i+z_i)^n.$$

Now choose $\boldsymbol{\alpha} = \mathbf{1}$ (length n) and $\boldsymbol{\beta} = \mathbf{1}$ (length $n-1$). We have

$$\begin{aligned}
\mathrm{Cap}_{(\mathbf{1},n\cdot\mathbf{1}-\mathbf{1})}(\mathrm{Symb}^{n\cdot\mathbf{1}}[T]) &= \inf_{\boldsymbol{x},\boldsymbol{z}>0} \frac{n\cdot z_n^{n-1}\prod_{i=1}^{n-1}(x_i+z_i)^n}{x_1\cdots x_{n-1}z_1^{n-1}\cdots z_n^{n-1}} \\
&= n\cdot \inf_{\boldsymbol{x},\boldsymbol{z}>0}\prod_{i=1}^{n-1}\frac{(x_i+z_i)^n}{x_i z_i^{n-1}} \\
&= n\cdot\prod_{i=1}^{n-1}\mathrm{Cap}_{(1,n-1)}\Big((x_i+z_i)^n\Big) \\
&= n\left[n\left(\frac{n}{n-1}\right)^{n-1}\right]^{n-1} = n\left[\frac{n^n}{(n-1)^{n-1}}\right]^{n-1}.
\end{aligned}$$

Last step: Our favorite calculus problem, raised to the $n-1$ power.

Sanity check: Gurvits' theorem

Last slide: For $T = \partial_{x_n}|_{x_n=0}$ and $\boldsymbol{\lambda} = n \cdot \mathbf{1}$, $\boldsymbol{\alpha} = \mathbf{1}$, $\boldsymbol{\beta} = \mathbf{1}$, we have

$$\mathrm{Cap}_{(\mathbf{1}, n\cdot\mathbf{1}-\mathbf{1})}(\mathrm{Symb}^{n\cdot\mathbf{1}}[T]) = n\left[\frac{n^n}{(n-1)^{n-1}}\right]^{n-1}.$$

By the theorem, we have

$$\frac{\mathrm{Cap}_{\mathbf{1}}(T[p])}{\mathrm{Cap}_{\mathbf{1}}(p)} \geq \left(\prod_{i=1}^{n} \frac{1^1(n-1)^{n-1}}{n^n}\right) n\left[\frac{n^n}{(n-1)^{n-1}}\right]^{n-1}$$
$$= n \cdot \frac{(n-1)^{n-1}}{n^n} = \left(\frac{n-1}{n}\right)^{n-1}.$$

Rearrange to get Gurvits' theorem:

$$\mathrm{Cap}_{\mathbf{1}}\left(\partial_{x_n}|_{x_n=0}\, p\right) \geq \left(\frac{n-1}{n}\right)^{n-1} \mathrm{Cap}_{\mathbf{1}}(p).$$

Bonus: Choosing more refined $\boldsymbol{\lambda}$ gives a better bound.

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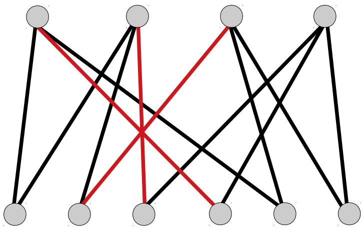
Potential application: Non-perfect matchings

Let G be a (a, b)-biregular (m, n)-bipartite graph $(am = bn)$ and consider:



Bipartite adjacency matrix, A:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$\#$ k-matchings $=$ k-subpermanents

(a, b)-regular $\iff$ (a, b)-stochastic

We can associate a polynomial to G in the same way as before:

$$p_G(\mathbf{x}) := \prod_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_j$$

Question: What operator picks out k-matchings?

Potential application: Non-perfect matchings

For (a, b)-regular (m, n)-bipartite G with adjacency matrix A:

$$p_G(\boldsymbol{x}) := \prod_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_j$$

$\#$ perfect matchings $=$ permanent $= \partial_{\boldsymbol{x}}^{[n]}\Big|_{\boldsymbol{x}=\boldsymbol{0}} p_G$

$\#$ k-matchings $=$ sum of k-subpermanents $\approx \sum_{S \in \binom{[n]}{k}} \partial_{\boldsymbol{x}}^{S}\Big|_{\boldsymbol{x}=\boldsymbol{1}} p_G$

Evaluation at 1 instead of 0: requires regularity of the graph.

$$\sum_{S \in \binom{[n]}{k}} \partial_{\boldsymbol{x}}^{S}\Big|_{\boldsymbol{x}=\boldsymbol{1}} p_G = \mu_k(G) \cdot a^{m-k} \qquad (a \text{ is the row sum})$$

Also: Degree of p_G is $\boldsymbol{\lambda} = (\text{col sum}) \cdot \boldsymbol{1} = (b, b, \ldots, b)$, by regularity.

Questions: Does $\sum_{S \in \binom{[n]}{k}} \partial_{\boldsymbol{x}}^{S}\Big|_{\boldsymbol{x}=\boldsymbol{1}}$ have real stable symbol? **Yes.** What is its capacity? **We will compute this.**

The counting operator

Last slide: $T = \sum_{S \in \binom{[n]}{k}} \partial_{\boldsymbol{x}}^S \Big|_{\boldsymbol{x}=\mathbf{1}}$ where $\boldsymbol{\lambda} = b \cdot \mathbf{1}$:

$$\begin{aligned}
\mathrm{Symb}^{\boldsymbol{\lambda}}[T](\boldsymbol{x}, \boldsymbol{z}) &= T\left[\prod_{i=1}^{n} (x_i + z_i)^b\right] \\
&= b^k \left[\sum_{S \in \binom{[n]}{n-k}} \prod_{i \in S} (1 + z_i)\right] \prod_{i=1}^{n} (1 + z_i)^{b-1} \\
&= b^k \cdot e_{n-k}(1 + z_1, \ldots, 1 + z_n) \cdot \prod_{i=1}^{n} (1 + z_i)^{b-1}.
\end{aligned}$$

This is a product of real stable polynomials $\implies$ $\mathrm{Symb}^{\boldsymbol{\lambda}}[T]$ is real stable.

What about capacity? Recall the theorem:

$$\frac{\mathrm{Cap}_{\boldsymbol{\beta}}(T[p_G])}{\mathrm{Cap}_{\boldsymbol{\alpha}}(p_G)} \geq \frac{\boldsymbol{\alpha}^{\boldsymbol{\alpha}} (\boldsymbol{\lambda} - \boldsymbol{\alpha})^{\boldsymbol{\lambda} - \boldsymbol{\alpha}}}{\boldsymbol{\lambda}^{\boldsymbol{\lambda}}} \cdot \mathrm{Cap}_{(\boldsymbol{\beta}, \boldsymbol{\lambda} - \boldsymbol{\alpha})}\left(\mathrm{Symb}^{b \cdot \mathbf{1}}[T]\right).$$

Know: $\boldsymbol{\beta}$ must be $\mathbf{0}$. **What about $\boldsymbol{\alpha}$?**

Choosing α based on G

Know: $\mathrm{Cap}_{\boldsymbol{\alpha}}(p) = p(\mathbf{1})$ iff $\boldsymbol{\alpha} = \nabla \log p(\mathbf{1})$. **What is $\nabla \log p_G(\mathbf{1})$?**

$$\nabla|_{\mathbf{x}=\mathbf{1}} \log \prod_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_j = \sum_{i=1}^{m} \nabla|_{\mathbf{x}=\mathbf{1}} \log \sum_{j=1}^{n} A_{ij} x_j = \sum_{i=1}^{m} \left(\frac{A_{ij}}{a} \right)_{j=1}^{n} = \frac{b}{a} \cdot \mathbf{1}.$$

So: $\mathrm{Cap}_{\frac{b}{a}\cdot 1}(p_G) = p_G(\mathbf{1}) = \prod_{i=1}^{m} a = a^m$.

Therefore:

$$\frac{a^{m-k} \cdot \mu_k(G)}{a^m} = \frac{\mathrm{Cap}_{\mathbf{0}}(T[p_G])}{\mathrm{Cap}_{\frac{b}{a}\cdot\mathbf{1}}(p_G)}$$

$$\geq \left(\frac{(\frac{b}{a})^{\frac{b}{a}} (b - \frac{b}{a})^{b - \frac{b}{a}}}{b^b} \right)^n \cdot \mathrm{Cap}_{(\mathbf{0}, (b - \frac{b}{a})\cdot\mathbf{1})} \left(\mathrm{Symb}^{b\cdot\mathbf{1}}[T] \right).$$

Note: $(b - \frac{b}{a}) \cdot \mathbf{1}$ does **not** pick out a coefficient of $\mathrm{Symb}^{b\cdot\mathbf{1}}[T]$.

Computing the capacity

Fingers crossed: Our choice of α works well with $\mathrm{Symb}[T]$ too.

$$\mathrm{Cap}_{(\mathbf{0},(b-\frac{b}{a})\cdot\mathbf{1})}\left(\mathrm{Symb}^{b\cdot\mathbf{1}}[T]\right) = \inf_{\boldsymbol{z}>0} \frac{b^k \cdot e_{n-k}(\mathbf{1}+\boldsymbol{z}) \cdot \prod_{i=1}^n (1+z_i)^{b-1}}{z_1^{b-\frac{b}{a}} \cdots z_n^{b-\frac{b}{a}}}.$$

Can we compute? $\mathrm{Symb}[T]$ and α symmetric $\implies$ inf attained at $z \cdot \mathbf{1}$:

$$\mathrm{Cap}_{(\mathbf{0},(b-\frac{b}{a})\cdot\mathbf{1})}\left(\mathrm{Symb}^{b\cdot\mathbf{1}}[T]\right) = \inf_{z>0} \frac{b^k \binom{n}{n-k} \cdot (1+z)^{n-k+n(b-1)}}{z^{n(b-\frac{b}{a})}}$$

This is our favorite calculus problem. **Therefore:**

$$\mu_k(G) \geq \text{explicit constant depending on } a, b, k, n.$$

This yields the best known bound at this level of generality. (Proven before by Csikvári using different "entropic" methods.)

Putting it all together

Theorem (Csikvári '14, Gurvits-L '18)

Given an (a, b)-regular bipartite graph on $m + n$ vertices, we have

$$\mu_k(G) \geq \binom{n}{k}(ab)^k \frac{m^m(ma-k)^{ma-k}}{(ma)^{ma}(m-k)^{m-k}}.$$

- **Pro:** Proof is elementary, after assuming the linear operator bound.
- **Con:** Proof was very delicate based on all parameters. **Though:** while row sum $= a$ must hold, we could choose $\boldsymbol{\lambda}$ based on vertex degrees in G and then estimate $\mathrm{Cap}(\mathrm{Symb}[T])$ via convex program.
- **???:** Csikvári's proof relied heavily on intuition about bipartite graphs; our proof does not at all.

[Schrijver '98]: Perfect matchings for d-regular bipartite graph for $m = n$.

$$\mu_n(G) \geq d^{2n} \frac{n^n(nd-n)^{nd-n}}{(nd)^{nd}} = \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^n.$$

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Potential application: Intersection of two matroids

Recall: A **matroid** M on a ground set E can be defined as a non-empty collection $\mathcal{B}$ of size-d subsets of E called **bases** of M, which satisfy:

- **Exchange axiom:** For all $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$, there exists $f \in B_2 \setminus B_1$ such that $B_1 \cup \{f\} \setminus \{e\} \in \mathcal{B}$.

E.g.: Linear bases in a collection of vectors, spanning trees of a graph.

Recall: For any matroid, the basis-generating polynomial is Lorentzian:

$$p_M(\boldsymbol{x}) := \sum_{B \in \mathcal{B}} \boldsymbol{x}^B = \sum_{B \in \mathcal{B}} \prod_{e \in B} x_e \in \mathbb{R}_+^{\mathbf{1}}[\boldsymbol{x}].$$

Matroid intersection problem: Given matroids M_1, M_2 on the same ground set E, count or approximate $|\mathcal{B}_1 \cap \mathcal{B}_2|$.

Answer: $|\mathcal{B}_1 \cap \mathcal{B}_2| = \langle p_{M_1}, p_{M_2} \rangle_{\mathbf{1}} \geq \boldsymbol{\alpha}^{\boldsymbol{\alpha}}(\mathbf{1}-\boldsymbol{\alpha})^{\mathbf{1}-\boldsymbol{\alpha}} \operatorname{Cap}_{\boldsymbol{\alpha}}(p_{M_1}) \operatorname{Cap}_{\boldsymbol{\alpha}}(p_{M_2})$.

New problem: How can we compute $\operatorname{Cap}_{\boldsymbol{\alpha}}(p_M)$? E.g., $p_M(\mathbf{1}) = |\mathcal{B}|$.

Caveats

Note: Above we said

$$|\mathcal{B}_1 \cap \mathcal{B}_2| = \langle p_{M_1}, p_{M_2} \rangle \geq \boldsymbol{\alpha}^{\boldsymbol{\alpha}}(\mathbf{1}-\boldsymbol{\alpha})^{\mathbf{1}-\boldsymbol{\alpha}} \operatorname{Cap}_{\boldsymbol{\alpha}}(p_{M_1}) \operatorname{Cap}_{\boldsymbol{\alpha}}(p_{M_2}).$$

Here we use the "untwisted" inner product $\langle p, q \rangle_{\mathbf{1}} = \sum_{\boldsymbol{\mu}} p_{\boldsymbol{\mu}} q_{\boldsymbol{\mu}}$.

How? Let $\tilde{q} := \boldsymbol{x}^{\mathbf{1}} \cdot q(x_1^{-1}, \ldots, x_n^{-1})$ for $q \in \mathbb{R}_+^{\mathbf{1}}[\boldsymbol{x}]$, which is real stable whenever q is real stable. Then:

$$\sum_{\boldsymbol{\mu}} p_{\boldsymbol{\mu}} q_{\boldsymbol{\mu}} = \sum_{\boldsymbol{\mu}} p_{\boldsymbol{\mu}} \tilde{q}_{\mathbf{1}-\boldsymbol{\mu}} \geq \boldsymbol{\alpha}^{\boldsymbol{\alpha}}(\mathbf{1}-\boldsymbol{\alpha})^{\mathbf{1}-\boldsymbol{\alpha}} \cdot \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\boldsymbol{\alpha}}} \cdot \inf_{\boldsymbol{x}>0} \frac{\tilde{q}(\boldsymbol{x})}{\boldsymbol{x}^{\mathbf{1}-\boldsymbol{\alpha}}}$$

Finally: $\displaystyle \inf_{\boldsymbol{x}>0} \frac{\tilde{q}(\boldsymbol{x})}{\boldsymbol{x}^{\mathbf{1}-\boldsymbol{\alpha}}} = \inf_{\boldsymbol{x}>0} \frac{\boldsymbol{x}^{\mathbf{1}} \cdot q(\boldsymbol{x}^{-1})}{\boldsymbol{x}^{\mathbf{1}-\boldsymbol{\alpha}}} = \inf_{\boldsymbol{x}>0} \frac{q(\boldsymbol{x})}{\boldsymbol{x}^{\boldsymbol{\alpha}}} = \operatorname{Cap}_{\boldsymbol{\alpha}}(q).$

Also: We use the real stable bound, even though matroid generating polynomials are only Lorentzian.

Further: The operation $\tilde{q}$ doesn't preserve Lorentzian! But it does preserve matroid generating polynomials.

A probabilistic approach

Associate a polynomial $f \in \mathbb{R}_+^{\mathbf{1}}[\boldsymbol{x}]$ to probability distribution $\boldsymbol{\nu}$ on $\text{supp}(f)$:

$$\mathbb{P}[\boldsymbol{\nu} = S] := f_S \quad \text{where} \quad f(\boldsymbol{x}) = \sum_{S \in \text{supp}(p)} f_S \boldsymbol{x}^S \quad \text{and} \quad f(\mathbf{1}) = 1.$$

Some facts:

- **Marginals** $\mathbb{E}[\boldsymbol{\nu}] = \nabla f(\mathbf{1}) =: \boldsymbol{\gamma}$. **How?**

$$\mathbb{E}[\nu_i] = \sum_S \delta_{i \in S} \cdot f_S = \sum_{S \ni i} f_S = \partial_{x_i} f(\mathbf{1}).$$

- **Entropy** $= \mathcal{H}(\boldsymbol{\nu}) = -\sum_S f_S \log f_S$. **Basic fact:**

$$\mathcal{H}(\boldsymbol{\nu}) \leq \sum_{i=1}^n \mathcal{H}(\gamma_i, 1 - \gamma_i) := \sum_{i=1}^n -\left[\gamma_i \log \gamma_i + (1 - \gamma_i) \log(1 - \gamma_i)\right].$$

- **[A-OG-V '18]:** If f log-concave in $\mathbb{R}_+^n$, then $\mathcal{H}(\boldsymbol{\nu}) \geq -\sum_{i=1}^n \gamma_i \log \gamma_i$.

Entropy proofs: Jensen's inequality, concavity/monotonicity of log, etc.

Probability and counting

For any collection $\mathcal{S}$ of subsets of $\{0,1\}^n$ (e.g. bases of matroid), consider:

$$f(\boldsymbol{x}) = \frac{1}{|\mathcal{S}|} \sum_{S \in \mathcal{S}} \boldsymbol{x}^S \quad \implies \quad \mathbb{P}[\boldsymbol{\nu} = S] = \frac{1}{|\mathcal{S}|}.$$

Compute the entropy: $\mathcal{H}(\boldsymbol{\nu}) = -\sum_S \frac{1}{|\mathcal{S}|} \log \frac{1}{|\mathcal{S}|} = \log |\mathcal{S}|$.

That is: We can approximate $|\mathcal{S}|$ by approximating the entropy of $\boldsymbol{\nu}$.

Apply entropy facts from the previous slide: If f log-concave,

$$\sum_{i=1}^n \mathcal{H}(\gamma_i, 1-\gamma_i) \geq \mathcal{H}(\boldsymbol{\nu}) = \log |\mathcal{S}| \geq -\sum_{i=1}^n \gamma_i \log \gamma_i.$$

Problems: How to approximate γ? (Just as hard as approximating $|\mathcal{S}|$.) We want to count $\mathcal{B}_1 \cap \mathcal{B}_2$, for which the polynomial is **not** log-concave.

Solutions: Capacity is related to the marginals $\nabla p(\mathbf{1})$. Use the inner product to relate individual $\mathcal{B}_i$ polynomials to the quantity $|\mathcal{B}_1 \cap \mathcal{B}_2|$.

Using the inner product bound

Given matroids M_1 and M_2 and any $\boldsymbol{\alpha}$, the inner product bounds gives

$$\begin{aligned}\log|\mathcal{B}_1 \cap \mathcal{B}_1| = \log\langle p_{M_1}, p_{M_2}\rangle_{\mathbf{1}} &\geq \log\left[\boldsymbol{\alpha}^{\boldsymbol{\alpha}}(\mathbf{1}-\boldsymbol{\alpha})^{\mathbf{1}-\boldsymbol{\alpha}}\operatorname{Cap}_{\boldsymbol{\alpha}}(p_{M_1})\operatorname{Cap}_{\boldsymbol{\alpha}}(p_{M_2})\right] \\ &= -\sum_{i=1}^n \mathcal{H}(\alpha_i, 1-\alpha_i) + \log\operatorname{Cap}_{\boldsymbol{\alpha}}(p_{M_1}) + \log\operatorname{Cap}_{\boldsymbol{\alpha}}(p_{M_2})\end{aligned}$$

In turns out (next week): When p is log-concave and $\boldsymbol{\alpha} \in \operatorname{Newt}(p)$, $\log\operatorname{Cap}_{\boldsymbol{\alpha}}(p)$ is the entropy of a log-concave distribution with marginals $\boldsymbol{\alpha}$.

Use our entropy facts: For any $\boldsymbol{\alpha} \in \operatorname{Newt}(p_{M_1}) \cap \operatorname{Newt}(p_{M_2})$,

$$\begin{aligned}\sum_{i=1}^n \mathcal{H}(\gamma_i, 1-\gamma_i) \geq \log|\mathcal{B}_1 \cap \mathcal{B}_2| &\geq -\sum_{i=1}^n \mathcal{H}(\alpha_i, 1-\alpha_i) - 2\sum_{i=1}^n \alpha_i \log\alpha_i \\ &= \sum_{i=1}^n \mathcal{H}(\alpha_i, 1-\alpha_i) + 2\sum_{i=1}^n (1-\alpha_i)\log(1-\alpha_i)\end{aligned}$$

Also: $-\sum_{i=1}^n (1-\alpha_i)\log(1-\alpha_i) \leq \sum_{i=1}^n \alpha_i = d = \text{rank/degree}.$

Putting it all together

Combine the above:

- f is the normalized generating polynomial for $\mathcal{B}_1 \cap \mathcal{B}_2$ with associated distribution $\boldsymbol{\nu}$ and marginals $\gamma \in \mathrm{Newt}(f)$.
- The entropy of $\boldsymbol{\nu}$ is given by $\mathcal{H}(\boldsymbol{\nu}) = \log |\mathcal{B}_1 \cap \mathcal{B}_2|$.
- p_1 and p_2 are the unnormalized generating polynomials for $\mathcal{B}_1$ and $\mathcal{B}_2$.
- $\boldsymbol{\alpha}$ is any element of $\mathrm{Newt}(p_1) \cap \mathrm{Newt}(p_2) = \mathrm{Newt}(f)$.

With all that, we have:

$$\sum_{i=1}^{n} \mathcal{H}(\gamma_i, 1-\gamma_i) \geq \log |\mathcal{B}_1 \cap \mathcal{B}_2| = \langle p_{M_1}, p_{M_2} \rangle_{\mathbf{1}} \geq \sum_{i=1}^{n} \mathcal{H}(\alpha_i, 1-\alpha_i) - 2d.$$

Finally: How do we actually approximate $\log |\mathcal{B}_1 \cap \mathcal{B}_2|$?

Answer: Choose $\boldsymbol{\alpha} \in \mathrm{Newt}(f)$ so that right-hand side is maximized.

$$\sum_{i=1}^{n} \mathcal{H}(\alpha_i, 1-\alpha_i) \geq \sum_{i=1}^{n} \mathcal{H}(\gamma_i, 1-\gamma_i) \geq \log |\mathcal{B}_1 \cap \mathcal{B}_2| \geq \sum_{i=1}^{n} \mathcal{H}(\alpha_i, 1-\alpha_i) - 2d.$$

The algorithm

Algo: Find α which maximizes $\sum_i \mathcal{H}(\alpha_i, 1-\alpha_i)$ over Newt(f), where

$$f(\boldsymbol{x}) = \frac{1}{|\mathcal{B}_1 \cap \mathcal{B}_2|} \sum_{B \in \mathcal{B}_1 \cap \mathcal{B}_2} \boldsymbol{x}^B.$$

How? First, $\text{Newt}(f) = \text{Newt}(p_1) \cap \text{Newt}(p_2)$. Next, matroid basis polytopes **characterize** collections of subsets for which the greedy algo (Kruskal) can be used to maximize linear functionals (need indep oracle).

Therefore:

1. Maximize linear functionals on $\text{Newt}(p_i) \implies$
2. Separation oracle for $\text{Newt}(p_i) \implies$
3. Separation oracle for $\text{Newt}(f) = \text{Newt}(p_1) \cap \text{Newt}(p_2)$
4. Efficient convex optimization for Newt(f) via ellipsoid method.

Now what? $-\sum_i \mathcal{H}(\alpha_i, 1-\alpha_i) = \sum_i [\alpha_i \log \alpha_i + (1-\alpha_i)\log(1-\alpha_i)]$ is a convex function on $[0,1]^n \implies$ Efficient algo for basis intersection.

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   - The algorithm
4. Open questions

Open questions

General open question: Are there other linear operators or situations where the inner product or linear operator capacity bounds can be used to bound or approximate some quantity?

As of now, only differential operators have been used and analyzed.

What situations require other operators?

Open problem: Given d-homogeneous real stable (or Lorentzian? or something more general?) polynomials $p, q \in \mathbb{R}_+[\boldsymbol{x}]$, define

$$\langle p, q \rangle_{\mathsf{SU}_n} := \sum_{|\boldsymbol{\mu}|=d} \binom{d}{\boldsymbol{\mu}}^{-1} p_{\boldsymbol{\mu}} q_{\boldsymbol{\mu}}$$

where $\binom{d}{\boldsymbol{\mu}}$ is the multinomial coefficient. Prove for any $\boldsymbol{\alpha}$ that:

$$\langle p, q \rangle_{\mathsf{SU}_n} \geq \frac{\alpha_1^{\alpha_1} \cdots \alpha_n^{\alpha_n}}{d^d} \cdot \mathrm{Cap}_{\boldsymbol{\alpha}}(p) \cdot \mathrm{Cap}_{\boldsymbol{\alpha}}(q).$$

Open even for p, q of the form $\det(\sum_i x_i A_i)$ for PSD A_i.