Capacity and Gurvits' Theorem Exercises

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Definition. Given a polynomial $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ and $\boldsymbol{\alpha} \in \mathbb{R}_+^n$, we define

$$\mathrm{Cap}_{\boldsymbol{\alpha}}(p) := \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\boldsymbol{\alpha}}}.$$

Theorem (Gurvits' theorem)**.** *If $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ is n-homogeneous and real stable, then*

$$\mathrm{Cap}_{\mathbf{1}}\left(\partial_{x_n}|_{x_n=0}\, p\right) \geq \left(\frac{n-1}{n}\right)^{n-1} \mathrm{Cap}_{\mathbf{1}}(p).$$

Theorem (Gurvits' corollary)**.** *If $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ is n-homogeneous and real stable, then*

$$p_{\mathbf{1}} = \partial_{x_1} \cdots \partial_{x_n} p \geq \frac{n!}{n^n} \mathrm{Cap}_{\mathbf{1}}(p).$$

Exercises

1. Prove the last step of the proof of Gurvits' theorem: For all real-rooted $p \in \mathbb{R}_+^d[t]$, we have that

$$p_1 = \partial_t p(0) \geq \left(\frac{d-1}{d}\right)^{d-1} \mathrm{Cap}_1(p) = \left(\frac{d-1}{d}\right)^{d-1} \inf_{t>0} \frac{p(t)}{t}.$$

   Hint: If $p(t) = \prod_{i=1}^d (s_i t + 1)$, then $p_1 = \sum_{i=1}^d s_i$. Then use the AM-GM inequality.

2. Use Exercise 1 to prove a refined form of Gurvits' theorem which depends on the degree of x_n in the polynomial p. Then use this to prove Schrijver's inequality: For any d-regular bipartite graph on $2n$ vertices, the number of perfect matchings of the graph is bounded via

$$\#\mathrm{pm}(G) \geq \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^n.$$

3. Prove that the permanent of any doubly stochastic matrix is at most 1.

4. Let $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ be a d-homogeneous polynomial such that $p(\mathbf{1}) = 1$ and $\nabla p(\mathbf{1}) = \boldsymbol{\mu} \in \mathbb{Z}_+^n$. Use Gurvits' corollary to prove

$$p_{\boldsymbol{\mu}} \geq \frac{d!}{d^d} \cdot \frac{\boldsymbol{\mu}^{\boldsymbol{\mu}}}{\boldsymbol{\mu}!} = \frac{d!}{d^d} \cdot \frac{\mu_1^{\mu_1} \cdots \mu_n^{\mu_n}}{\mu_1! \cdots \mu_n!}.$$

   Hint: Consider the polynomial

$$q(\boldsymbol{y}) := p\left(\frac{y_{1,1} + \cdots + y_{1,\mu_1}}{\mu_1}, \frac{y_{2,1} + \cdots + y_{2,\mu_2}}{\mu_2}, \ldots, \frac{y_{n,1} + \cdots + y_{n,\mu_n}}{\mu_n}\right),$$

   which is d-homogeneous in $\mu_1 + \cdots + \mu_n$ variables.