Capacity Bounds on Polynomial Coefficients

Polynomial Capacity: Theory, Applications, Generalizations

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Notation

Polynomial notation:

- $\mathbb{R}, \mathbb{R}_+, \mathbb{Z}_+ :=$ reals, non-negative reals, non-negative integers.
- $\boldsymbol{x}^{\boldsymbol{\mu}} := \prod_i x_i^{\mu_i}$ and $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ is entrywise.
- $\mathbb{R}[\boldsymbol{x}] :=$ v.s. of real polynomials in n variables.
- $\mathbb{R}_+[\boldsymbol{x}] :=$ v.s. of real polynomials with non-negative coefficients.
- $\mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}] :=$ v.s. of polynomials of degree at most λ_i in x_i.
- For $p \in \mathbb{R}[\boldsymbol{x}]$, we write $p(\boldsymbol{x}) = \sum_{\boldsymbol{\mu}} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$.
- For d-homogeneous $p \in \mathbb{R}[\boldsymbol{x}]$, we write $p(\boldsymbol{x}) = \sum_{|\boldsymbol{\mu}|=d} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$.
- $\frac{d}{dx} = \frac{\partial}{\partial x} = \partial_x :=$ derivative with respect to x, and $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} := \prod_i \partial_{x_i}^{\mu_i}$.
- $\text{supp}(p) =$ **support** of $p =$ the set of $\boldsymbol{\mu} \in \mathbb{Z}_+^n$ for which $p_{\boldsymbol{\mu}} \neq 0$.
- $\text{Newt}(p) =$ **Newton polytope** of $p =$ convex hull of the support of p as a subset of $\mathbb{R}^n$.

Recall: The big three

The **geometry of polynomials** is generally an investigation of the connections between the various properties of polynomials:

- **Algebraic**, via the roots/zeros of the polynomial.
- **Combinatorial**, via the coefficients of the polynomial.
- **Analytic**, via the evaluations of the polynomial.

Why do we care? We use features of the interplay between these three to prove facts about mathematical objects which a priori have nothing to do with polynomials.

Typical method:

1. Encode some object as a polynomial which has some nice properties.
2. Apply operations to that polynomial which preserve those properties.
3. **Extract information** at the end which relates back to the object.

Outline

1. Coefficient bounds via capacity
   - General bound for univariate polynomials
   - Generalization to multivariate polynomials
   - Bounds for various polynomial classes
2. Application: Mixed discriminant and mixed volume
   - Relation to the permanent via polarization
   - Capacity bounds
   - Counting solutions to polynomial systems over $\mathbb{C}$
3. Application: Counting contingency tables
   - The generating polynomial for contingency tables
   - Capacity bounds
   - Volume of the Birkhoff polytope

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Univariate coefficient bound

Last time: Gurvits' bound on $p_{\mathbf{1}}$ for n-homogeneous $p \in \mathbb{R}_+[x_1, \ldots, x_n]$:

$$\mathrm{Cap}_{\mathbf{1}}(p) \geq p_{\mathbf{1}} \geq \frac{n!}{n^n} \mathrm{Cap}_{\mathbf{1}}(p) \quad \text{where} \quad \mathrm{Cap}_{\alpha}(p) := \inf_{\boldsymbol{x} > \mathbf{0}} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\alpha}}.$$

Missing piece of the proof: Coefficient bound for univariate polynomials.

Lemma (Brändén-L-Pak '20)

Let $q, w \in \mathbb{R}_+^d[t]$ be such that $\left(\frac{q_k}{w_k}\right)_{k=0}^{d}$ forms a log-concave sequence. For all $k \in \{0, \ldots, d\}$, if $q_k > 0$ then

$$\frac{q_k}{\mathrm{Cap}_k(q)} \geq \frac{w_k}{\mathrm{Cap}_k(w)} \iff \mathrm{Cap}_k(w) \geq \frac{w_k}{q_k} \cdot \mathrm{Cap}_k(q).$$

Equivalent: $\mathrm{Cap}_k(w) = \sup_{\boldsymbol{a} \text{ log-concave}} \left[\inf_{x>0} \frac{\sum_{j=0}^{d} a_j w_j x^j}{a_k x^k} \right]$ via $a_k = \frac{q_k}{w_k}$.

Proof of the bound

Lemma: $\mathrm{Cap}_k(w) = \sup_{\boldsymbol{a}\text{ log-concave}} \left[\inf_{x>0} \frac{\sum_{j=0}^d a_j w_j x^j}{a_k x^k} \right] =: C_k.$

Proof: WLOG $a_k = 1$, which gives

$$C_k = \sup_{\substack{\boldsymbol{a}\text{ log-concave} \\ a_k=1}} \inf_{x>0} \left[\left(\sum_{j=0}^{k-1} a_j w_j x^{j-k} \right) + w_k + \left(\sum_{j=k+1}^{d} a_j w_j x^{j-k} \right) \right].$$

Log-concavity $\implies$ $a_{k-j} = a_k^{j-1} a_{k-j} \leq a_{k-1}^j$ and $a_{k+j} = a_k^{j-1} a_{k+j} \leq a_{k+1}^j$, which implies $\boldsymbol{a}' := (a_{k-1}^k, \ldots, a_{k-1}, 1, a_{k+1}, \ldots, a_{k+1}^{d-k}) \geq \boldsymbol{a}$.

Further: $\boldsymbol{a}'' := (a_{k+1}^{j-k})_{j=0}^d \geq \boldsymbol{a}'$ by forcing $a_{k-1} a_{k+1} = 1$. **Therefore:**

$$C_k = \sup_{a_{k+1}>0} \left[\inf_{x>0} \left(\sum_{j=0}^{d} a_{k+1}^{j-k} w_j x^{j-k} \right) \right] = \sup_{a>0} \left[\inf_{y=ax>0} \left(\sum_{j=0}^{d} w_j y^{j-k} \right) \right],$$

which implies $C_k = \mathrm{Cap}_k(w)$.

Examples of the univariate bound

Lemma: If $q, w \in \mathbb{R}_+^d[t]$ such that $\frac{q_0}{w_0}, \frac{q_1}{w_1}, \ldots, \frac{q_d}{w_d}$ is log-concave, then

$$q_k \geq \frac{w_k}{\mathrm{Cap}_k(w)} \cdot \mathrm{Cap}_k(q).$$

Corollary: If $p(x, y)$ is d-homogeneous CLC, then

$$p_k \geq \binom{d}{k} \frac{k^k (d-k)^{d-k}}{d^d} \cdot \mathrm{Cap}_k\left(p(x, 1)\right) \approx \sqrt{\frac{d}{2\pi k(d-k)}} \cdot \mathrm{Cap}_{(k,d-k)}(p).$$

Proof: CLC equivalent to ULC coefficients. For $w_k = \binom{d}{k}$, we have

$$\mathrm{Cap}_k\left((x+1)^d\right) = \inf_{x>0} \frac{(x+y)^d}{x^k} = \frac{d^d}{k^k (d-k)^{d-k}}.$$

Corollary: If $p(x)$ has log-concave coefficients, then

$$p_k \geq \frac{\mathrm{Cap}_k(p)}{\mathrm{Cap}_k(w)} \geq \frac{k^k}{(k+1)^{k+1}} \cdot \mathrm{Cap}_k(p) \geq \frac{1}{e(k+1)} \cdot \mathrm{Cap}_k(p).$$

Proof: $w(x) = 1 + x + \cdots + x^d \leq \frac{1}{1-x} \implies$ another calculus exercise.

Generalization to multivariate polynomials

Now: How do we go from univariate to multivariate? **Same as before:**

$$\boldsymbol{\mu}! \cdot p_{\boldsymbol{\mu}} = \partial_{x_1}^{\mu_1}\Big|_{x_1=0} \cdots \partial_{x_n}^{\mu_n}\Big|_{x_n=0} p \geq K(\boldsymbol{\mu}) \cdot \mathrm{Cap}_{\boldsymbol{\mu}}(p),$$

for some constant $K(\boldsymbol{\mu})$. **Next:** Determine per-variable bound

$$\mathrm{Cap}_{(\mu_1,\ldots,\mu_{n-1})}\left(\partial_{x_n}^{\mu_n}\Big|_{x_n=0} p\right) \geq K(\mu_n) \cdot \mathrm{Cap}_{\boldsymbol{\mu}}(p).$$

by fixing $x_1, \ldots, x_{n-1} > 0$ and proving

$$k! \cdot q_k = \partial_t^k q(0) \geq K(k) \cdot \mathrm{Cap}_k(q),$$

where $q(t) := p(x_1, \ldots, x_{n-1}, t)$. **So:** For all $x_1, \ldots, x_{n-1} > 0$, we have

$$\frac{\partial_t^k\Big|_{t=0} p(x_1, \ldots, x_{n-1}, t)}{x_1^{\mu_1} \cdots x_{n-1}^{\mu_{n-1}} t^{\mu_n}} \geq K(\mu_n) \cdot \inf_{t>0} \frac{p(x_1, \ldots, x_{n-1}, t)}{x_1^{\mu_1} \cdots x_{n-1}^{\mu_{n-1}} t^{\mu_n}}.$$

Taking inf over $x_1, \ldots, x_{n-1}$ then gives the multivariate bound.

Generalization to multivariate polynomials

Let $\mathcal{C}$ be some class of polynomials in $\mathbb{R}_+[\boldsymbol{x}]$. **What did we need for the above argument to go through?**

1. Class $\mathcal{C}$ must be preserved under $\partial_{x_i}^k\big|_{x_i=0}$ for all i, k.
2. Class $\mathcal{C}$ must be preserved under positive evaluations of x_i for all i.

E.g.: Real stable polynomials satisfy all these properties. **From before:**

$$q_k \geq \binom{d}{k} \frac{k^k (d-k)^{d-k}}{d^d} \operatorname{Cap}_k(q)$$

when $q \in \mathbb{R}_+^d[t]$ is real-rooted ($\implies$ ULC coefficients). **So:**

Theorem (Gurvits)

If $p \in \mathbb{R}_+^{\boldsymbol{\lambda}}[x_1, \ldots, x_n]$ is real stable, then for any $\boldsymbol{\mu} \in \mathbb{Z}_+$:

$$\operatorname{Cap}_{\boldsymbol{\mu}}(p) \geq p_{\boldsymbol{\mu}} \geq \left[\prod_{i=1}^n \binom{\lambda_i}{\mu_i} \frac{\mu_i^{\mu_i}(\lambda_i - \mu_i)^{\lambda_i - \mu_i}}{\lambda_i^{\lambda_i}}\right] \operatorname{Cap}_{\boldsymbol{\mu}}(p).$$

Generalization to Lorentzian polynomials

What about Lorentzian/CLC polynomials? **Problem:** Positive evaluations break homogeneity $\implies$ does **not** preserve Lorentzian/CLC.

Work-around: $q(t,s) := p(x_1 \cdot s, \ldots, x_{n-1} \cdot s, t)$ for $x_1, \ldots, x_{n-1} > 0$.

New issue: $\deg(q) = \deg(p)$ even if $\deg_{x_n}(p) < \deg(p) \implies$ we cannot refer to per-variable degree.

At least: By the same argument, we still get Gurvits' theorem for CLC n-homogeneous polynomials in n variables.

Theorem (Gurvits)

If $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ is d-homogeneous and Lorentzian/CLC, then:

$$\mathrm{Cap}_{\mathbf{1}}(p) \geq p_{\mathbf{1}} \geq \frac{n!}{n^n} \mathrm{Cap}_{\mathbf{1}}(p).$$

Generalization to Lorentzian polynomials

More: Given d-homogeneous CLC $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ and $\boldsymbol{\mu} \in \mathbb{Z}_+^n$, define

$$q(\boldsymbol{y}) := p\left(\frac{y_{1,1} + \cdots y_{1,\mu_1}}{\mu_1}, \ldots, \frac{y_{n,1} + \cdots y_{n,\mu_n}}{\mu_n}\right).$$

- Since p is d-homogeneous, then $\sum_{i=1}^n \mu_i = d \implies q$ is d-homogeneous in d variables.
- $q_{\mathbf{1}} = \frac{\mu_1! \cdots \mu_n!}{\mu_1^{\mu_1} \cdots \mu_n^{\mu_n}} p_{\boldsymbol{\mu}}$ by expanding $\left(\frac{y_{1,1}+\cdots y_{1,\mu_1}}{\mu_1}\right)^{\mu_1} \cdots \left(\frac{y_{n,1}+\cdots y_{n,\mu_n}}{\mu_n}\right)^{\mu_n}$.

Therefore: $p_{\boldsymbol{\mu}} \geq \frac{d!}{d^d} \cdot \frac{\mu_1^{\mu_1} \cdots \mu_n^{\mu_n}}{\mu_1! \cdots \mu_n!} \mathrm{Cap}_{\mathbf{1}}(q)$.

Corollary (Gurvits)

If $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ is d-homogeneous and CLC, then for any $\boldsymbol{\mu} \in \mathbb{Z}_+$:

$$\mathrm{Cap}_{\boldsymbol{\mu}}(p) \geq p_{\boldsymbol{\mu}} \geq \frac{d!}{d^d} \cdot \frac{\mu_1^{\mu_1} \cdots \mu_n^{\mu_n}}{\mu_1! \cdots \mu_n!} \mathrm{Cap}_{\boldsymbol{\mu}}(p).$$

Generalization to denormalized Lorentzian polynomials

A d-homogeneous polynomial $p \in \mathbb{R}_+[\boldsymbol{x}]$ is **denormalized Lorentzian** if

$$N[p] := \sum_{\boldsymbol{\mu}} \frac{p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}}{\boldsymbol{\mu}!} = \sum_{\boldsymbol{\mu}} \frac{1}{\mu_1! \cdots \mu_n!} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}} \quad \text{is Lorentzian.}$$

Properties:

- ∂_{x_i} on Lorentzian $\iff$ $\times \frac{1}{x_i}$ on denormalized Lorentzian.
- Unclear if ∂_{x_i} preserves, but $\partial_{x_i}^k \Big|_{x_i=0}$ does.
- Preserved under "variable division", unlike Lorentzian.
- Preserved under "homogeneous" evaluations > 0, via symbol theorem.
- Preserved under products, via symbol theorem.
- **Symbol thoerem:** T preserves iff $N \circ T \circ N^{-1}$ preserves CLC.

Examples:

- Schur polynomials [Huh-Matherne-Mészáros-Dizier '19].
- **Conjecture:** Schubert polys. [Huh-Matherne-Mészáros-Dizier '19].
- Contingency tables generating functions [Brändén-L-Pak '20].

Generalization to denormalized Lorentzian polynomials

A d-homogeneous polynomial $p \in \mathbb{R}_+[\boldsymbol{x}]$ is **denormalized Lorentzian** if

$$N[p] := \sum_{\boldsymbol{\mu}} \frac{p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}}{\boldsymbol{\mu}!} = \sum_{\boldsymbol{\mu}} \frac{1}{\mu_1! \cdots \mu_n!} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}} \quad \text{is Lorentzian.}$$

For bivariate d-homogeneous p, equivalent to $(p_0, \ldots, p_d)$ log-concave.

Set $w_j = 1$ for all j and use the lemma to get:

$$p_k \geq \frac{k^k}{(k+1)^{k+1}} \cdot \mathrm{Cap}_k(p) \geq \frac{1}{e(k+1)} \cdot \mathrm{Cap}_k(p).$$

Theorem

If $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ is denormalized Lorentzian, then for any $\boldsymbol{\mu} \in \mathbb{Z}_+^n$:

$$\mathrm{Cap}_{\boldsymbol{\mu}}(p) \geq p_{\boldsymbol{\mu}} \geq \left[\prod_{i=2}^{n} \frac{\mu_i^{\mu_i}}{(\mu_i + 1)^{\mu_i + 1}}\right] \mathrm{Cap}_{\boldsymbol{\mu}}(p).$$

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Constructing the mixed discriminant/volume

The **mixed discriminant** of $n \times n$ matrices $A_1, \ldots, A_n$ and the **mixed volume** of compact convex sets $K_1, \ldots, K_n \subset \mathbb{R}^n$ are given by:

$$\frac{1}{n!}\partial_{x_1} \cdots \partial_{x_n} \det\left(\sum_i x_i A_i\right) \quad \text{and} \quad \frac{1}{n!}\partial_{x_1} \cdots \partial_{x_n} \mathrm{vol}\left(\sum_i x_i K_i\right).$$

Recall: $\mathrm{per}(M) = \partial_{x_1} \cdots \partial_{x_n} \prod_i \left(\sum_j m_{ij} x_j\right)$. (Same, up to $n!$.)

Another way for mixed discriminant: Construct $P(A_1, \ldots, A_n)$ such that

1. P is symmetric in its entries.
2. P is multilinear.
3. $P(A, A, \ldots, A) = \det(A)$.

Then P is the mixed discriminant.

Permanent: Think of P as a function of the cols of M, and $\det \to \prod_i x_i$.

Mixed volume: Think of P as a function of the compact convex sets?

Capacity bound for the mixed discriminant

Mixed discriminant: For $n \times n$ matrices $A_1, \dots, A_n$,

$$D(A_1, \dots, A_n) := \frac{1}{n!} \partial_{x_1} \cdots \partial_{x_n} \det \left(\sum_{i=1}^n x_i A_i \right).$$

If $A_1, \dots, A_n$ are PSD, then $\det\left(\sum_{i=1}^n a_i A_i\right)$ is n-homogeneous real stable.

Corollary [Gurvits]: $n! \cdot D(A_1, \dots, A_n) \geq \frac{n!}{n^n} \operatorname{Cap}_{\mathbf{1}} \left[\det \left(\sum_{i=1}^n x_i A_i \right) \right].$

We can compute $\det\left(\sum_{i=1}^n x_i A_i\right)$ efficiently $\implies$ approximation algorithm.

Also: If $\operatorname{tr}(A_i) = 1$ for all i and $\sum_i A_i = I$, then $\boldsymbol{A}$ is a **doubly stochastic tuple** of matrices $\implies$ $\det\left(\sum_{i=1}^n x_i A_i\right)$ is a doubly stochastic polynomial.

Therefore: $n! \cdot D(A_1, \dots, A_n) \geq \frac{n!}{n^n}$ in this case.

Permanent: If $A_k = \operatorname{diag}(\boldsymbol{a}_k)$, then $n! \cdot D(A_1, \dots, A_n) = \operatorname{per} \begin{pmatrix} | & & | \\ \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_n \\ | & & | \end{pmatrix}.$

Capacity bound for the mixed volume

Mixed volume: For compact convex $K_1, \ldots, K_n \subset \mathbb{R}^n$,

$$V(K_1, \ldots, K_n) := \frac{1}{n!} \partial_{x_1} \cdots \partial_{x_n} \operatorname{vol}\left(\sum_{i=1}^n x_i K_i\right).$$

Exercise from before: $\operatorname{vol}\left(\sum_{i=1}^n a_i A_i\right)$ is n-homogeneous Lorentzian, by the Alexandrov-Fenchel inequalities.

Corollary [Gurvits]: $n! \cdot V(K_1, \ldots, K_n) \geq \frac{n!}{n^n} \operatorname{Cap}_{\mathbf{1}}\left[\operatorname{vol}\left(\sum_{i=1}^n x_i K_i\right)\right].$

Problem: How to compute $\operatorname{vol}\left(\sum_{i=1}^n x_i K_i\right)$ efficiently?

Deterministic algo: Not possible efficiently, by [Bárány-Füredi '87].

Randomized algo: Many options via weak membership oracle; current best is [Lovász-Vempala '06]. (I think?)

Therefore [Gurvits]: Randomized algo to compute the mixed volume.

Solutions to generic polynomial systems

Theorem [Bernstein–Khovanskii–Kushnirenko '75]: Given polynomials $f_1, \ldots, f_n$, the number of complex solutions to $f_1 = f_2 = \cdots = 0$ is equal to

$$V(\mathrm{Newt}(f_1), \ldots, \mathrm{Newt}(f_n)),$$

assuming the non-zero coefficients of $f_1, \ldots, f_n$ are generic.

Last slide: Randomized algo to approximate within factor e^{-n}:

- Membership oracle dependent on $\sum_i |\mathrm{supp}(f_i)|$.
- Polynomial-time randomized volume approximation for evaluation of the polynomial $\mathrm{vol}\left(\sum_i x_i \,\mathrm{Newt}(f_i)\right)$ for $\boldsymbol{x} > 0$.
- Ellipsoid method to compute capacity.

Question: Is this useful? Can this be done with more basic methods?

Question: What is a "doubly stochastic" (up to scalar) tuple $\mathrm{Newt}(\boldsymbol{f})$?
$\implies$ Explicit upper and lower bounds in this case.

More general coefficients and mixed forms

More general: Given $d \times d$ PSD matrices $A_1, \ldots, A_n$ with $d \geq n$, we have that $\det\left(\sum_{i=1}^n x_i A_i\right)$ is a d-homogeneous real stable polynomial.

Now: $D(A_1, \ldots, A_1, A_2, \ldots, A_2, \ldots)$ with μ_i copies of A_i for all i:

$$\binom{d}{\boldsymbol{\mu}} \cdot D(\boldsymbol{A}) \geq \frac{d!}{d^d} \cdot \frac{\mu_1^{\mu_1} \cdots \mu_n^{\mu_n}}{\mu_1! \cdots \mu_n!} \operatorname{Cap}_{\boldsymbol{\mu}} \left[\det\left(\sum_{i=1}^n x_i A_i\right)\right].$$

Equivalently: $D(\boldsymbol{A}) \geq \frac{\mu_1^{\mu_1} \cdots \mu_n^{\mu_n}}{d^d} \operatorname{Cap}_{\boldsymbol{\mu}} \left[\det\left(\sum_{i=1}^n x_i A_i\right)\right].$

Also: Same bound for mixed volume with μ_i copies of $K_i \subset \mathbb{R}^d$:

$$V(\boldsymbol{K}) \geq \frac{\mu_1^{\mu_1} \cdots \mu_n^{\mu_n}}{d^d} \operatorname{Cap}_{\boldsymbol{\mu}} \left[\operatorname{vol}\left(\sum_{i=1}^n x_i K_i\right)\right].$$

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Contingency tables

Given vectors $\boldsymbol{\alpha} \in \mathbb{Z}_+^m$ and $\boldsymbol{\beta} \in \mathbb{Z}_+^n$, a **contingency table** is a $m \times n$ matrix $M = (m_{ij})$ with $\mathbb{Z}_+$ entries such that

$$\sum_{i=1}^{m} m_{ij} = \beta_j \quad \text{for all } j \quad \text{and} \quad \sum_{j=1}^{n} m_{ij} = \alpha_i \quad \text{for all } i.$$

Definition: $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \#$ of contingency tables with "marginals" $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

E.g.: For $n = m$, $\mathrm{CT}(d \cdot \mathbf{1}, d \cdot \mathbf{1})$ is the number of (non-simple) d-regular bipartite graphs on $2n$ vertices. (Similar interpretation more generally.)

Generating function: $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) =$ coefficient $p_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ for

$$p^d(\boldsymbol{x}, \boldsymbol{y}) := \prod_{i=1}^{m} \prod_{j=1}^{n} \left(1 + x_i y_j + (x_i y_j)^2 + \cdots + (x_i y_j)^d\right)$$

where $d \geq \max(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n)$. **Why?** $(x_i y_j)^k \iff m_{ij} = k$.

Contingency tables generating function

Contingency tables generating function:

$$p^\infty(\boldsymbol{x}, \boldsymbol{y}) := \prod_{i=1}^m \prod_{j=1}^n \left(1 + x_i y_j + (x_i y_j)^2 + \cdots\right) = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}} \mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cdot \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{\boldsymbol{\beta}}.$$

Also: Max degree term of $(i, j)^{\text{th}}$ sum is max value of m_{ij}.

Problem: This fits into no class of polynomials we've looked at.

Solution: Invert the $\boldsymbol{y}$ variables for max degree d:

$$\tilde{p}^d(\boldsymbol{x}, \boldsymbol{y}) = \left[\prod_{j=1}^n y_j^{dm}\right] \cdot p^d(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^m \prod_{j=1}^n \left(y_j^d + x_i y_j^{d-1} + \cdots + x_i^d\right).$$

The $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ coefficient of p^d is the $(\boldsymbol{\alpha}, d \cdot \mathbf{1} - \boldsymbol{\beta})$ coefficient of $\tilde{p}^d$.

$\sum_{k=0}^d x_i^k y_j^{d-k}$ is denormalized Lorentzian, and products preserve denormalized Lorentzian $\implies$ $\tilde{p}^d$ is denormalized Lorentzian.

Capacity bounds for contingency tables

Recall: For denormalized Lorentzian $p \in \mathbb{R}_+[\boldsymbol{x}]$ and $\boldsymbol{\mu} \in \mathbb{Z}_+^n$, we have

$$\mathrm{Cap}_{\boldsymbol{\mu}}(p) \geq p_{\boldsymbol{\mu}} \geq \left[\prod_{i=2}^{n} \frac{\mu_i^{\mu_i}}{(\mu_i + 1)^{\mu_i+1}}\right] \mathrm{Cap}_{\boldsymbol{\mu}}(p) \geq \left[\prod_{i=2}^{n} \frac{1}{e(\mu_i + 1)}\right] \mathrm{Cap}_{\boldsymbol{\mu}}(p).$$

Therefore: Lower bound on $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$:

$$\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \geq e^{1-n-m} \prod_{i=2}^{m} \frac{1}{\alpha_i + 1} \prod_{j=1}^{n} \frac{1}{d - \beta_j + 1} \cdot \mathrm{Cap}_{(\boldsymbol{\alpha}, d\cdot\mathbf{1}-\boldsymbol{\beta})}(\tilde{p}^d).$$

Actually: We can make the bounds symmetric and simplify:

$$\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \geq e^{1-n-m} \prod_{i=2}^{m} \frac{1}{\alpha_i + 1} \prod_{j=1}^{n} \frac{1}{\beta_j + 1} \cdot \mathrm{Cap}_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}(p^{\infty}).$$

Bonus: We are counting lattice points in various polytopes. By scaling and limiting, we can achieve lower bounds on volumes of these polytopes.
E.g.: Birkhoff polytope, flow polytopes, transportation polytopes

Bounding the volume of the Birkhoff polytope

Birkhoff polytope: Matrices with ≥ 0 entries and row/col sums $= 1$.

Discrete approximation: Count contingency tables with $\boldsymbol{\alpha} = \boldsymbol{\beta} = d \cdot \mathbf{1}$ in $\mathbb{Z}_+^n$, and then divide by appropriate scaling factor and limit:

$$\begin{aligned}\mathrm{vol}(B_n) &= \lim_{d\to\infty} \frac{\mathrm{CT}(d\cdot\mathbf{1}, d\cdot\mathbf{1})}{d^{(n-1)^2}} \\ &\geq \lim_{d\to\infty} \frac{e^{1-2n}}{d^{n^2-2n+1}} \prod_{i=2}^n \frac{1}{d+1} \prod_{j=1}^n \frac{1}{d+1} \cdot \inf_{\boldsymbol{x},\boldsymbol{y}>0} \left[\frac{\prod_{i,j=1}^n \sum_{k=0}^d (x_i y_j)^k}{\boldsymbol{x}^{d\cdot\mathbf{1}} \boldsymbol{y}^{d\cdot\mathbf{1}}}\right] \\ &= \lim_{d\to\infty} e^{1-2n} \inf_{\boldsymbol{x},\boldsymbol{y}>0} \prod_{i,j=1}^n \left[\frac{1}{d} \sum_{k=0}^d (x_i y_j)^{k-\frac{d}{n}}\right].\end{aligned}$$

Map $x_i \to x_i^{\frac{1}{d}}$ and $y_j \to y_j^{\frac{1}{d}}$, and then $\displaystyle\frac{1}{d}\sum_{k=0}^d (x_i y_j)^{\frac{k}{d}-\frac{1}{n}} \approx \int_0^1 (x_i y_j)^{t-\frac{1}{n}} dt$.

Note: Factor of $d+1$ in our bound is exactly what is needed here.