More Capacity Bounds on Coefficients

Polynomial Capacity: Theory, Applications, Generalizations

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Notation

Polynomial notation:

- $\mathbb{R}, \mathbb{R}_+, \mathbb{Z}_+ :=$ reals, non-negative reals, non-negative integers.
- $\boldsymbol{x}^{\boldsymbol{\mu}} := \prod_i x_i^{\mu_i}$ and $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ is entrywise.
- $\mathbb{R}[\boldsymbol{x}] :=$ v.s. of real polynomials in n variables.
- $\mathbb{R}_+[\boldsymbol{x}] :=$ v.s. of real polynomials with non-negative coefficients.
- $\mathbb{R}^{\boldsymbol{\lambda}}[\boldsymbol{x}] :=$ v.s. of polynomials of degree at most λ_i in x_i.
- For $p \in \mathbb{R}[\boldsymbol{x}]$, we write $p(\boldsymbol{x}) = \sum_{\boldsymbol{\mu}} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$.
- For d-homogeneous $p \in \mathbb{R}[\boldsymbol{x}]$, we write $p(\boldsymbol{x}) = \sum_{|\boldsymbol{\mu}|=d} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$.
- $\frac{d}{dx} = \frac{\partial}{\partial x} = \partial_x :=$ derivative with respect to x, and $\partial_{\boldsymbol{x}}^{\boldsymbol{\mu}} := \prod_i \partial_{x_i}^{\mu_i}$.
- $\text{supp}(p) =$ **support** of $p =$ the set of $\boldsymbol{\mu} \in \mathbb{Z}_+^n$ for which $p_{\boldsymbol{\mu}} \neq 0$.
- $\text{Newt}(p) =$ **Newton polytope** of $p =$ convex hull of the support of p as a subset of $\mathbb{R}^n$.

Recall: The big three

The **geometry of polynomials** is generally an investigation of the connections between the various properties of polynomials:

- **Algebraic**, via the roots/zeros of the polynomial.
- **Combinatorial**, via the coefficients of the polynomial.
- **Analytic**, via the evaluations of the polynomial.

Why do we care? We use features of the interplay between these three to prove facts about mathematical objects which a priori have nothing to do with polynomials.

Typical method:

1. Encode some object as a polynomial which has some nice properties.
2. Apply operations to that polynomial which preserve those properties.
3. **Extract information** at the end which relates back to the object.

Outline

1. So far in the course
   - Real stable polynomials
   - Lorentzian/CLC polynomials
   - Polynomial capacity
2. Coefficient bounds via capacity
   - Overview
   - Applications thus far
3. Computing capacity bounds on coefficients
   - Univariate bounds
   - Univariate bounds for real-rooted/Lorentzian polynomials
   - Multivariate bounds for real stable polynomials
4. Application to counting contingency tables
   - The generating polynomial for contingency tables
   - Capacity bounds for binary contingency tables
   - General contingency tables

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So far in the course: Real stable polynomials

Real stable polynomial $p \in \mathbb{R}[\boldsymbol{x}] = \mathbb{R}[x_1, \ldots, x_n]$

- **Definition:** $p(z_1, \ldots, z_n) \neq 0$ for all $z_i \in \mathcal{H}_+$ (upper half-plane).
- **Intuition:** Polynomials with log-concavity properties.
- **Intuition:** Generalizes real-rooted univariate polynomials, which have ultra log-concave coefficients ($\frac{p_k}{\binom{d}{k}}$ is a log-concave sequence). Also, **strong Rayleigh inequalities** are a crucial generalization.

Borcea-Brändén characterization of linear preservers

- Method for determining if a linear operator preserves real stability.
- Morally, T preserves stability iff its **symbol** does:

$$\mathrm{Symb}[T](\boldsymbol{x}, \boldsymbol{z}) := T\left[\prod_{i=1}^{n}(x_i + z_i)^{\lambda_i}\right] = \sum_{\boldsymbol{\mu} \leq \boldsymbol{\lambda}} \binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}} \boldsymbol{z}^{\boldsymbol{\lambda}-\boldsymbol{\mu}} T[\boldsymbol{x}^{\boldsymbol{\mu}}].$$

- **Intuition:** Apply the liner operator T to a "generic" polynomial.
- **E.g.:** $p|_{x_i=a}$ for $a \in \mathbb{R}$, $\nabla_{\boldsymbol{v}} p$ for $\boldsymbol{v} \in \mathbb{R}^n_+$, $p(A\boldsymbol{x})$ for A with ≥ 0 entries

So far in the course: Lorentzian / CLC polynomials

Lorentzian / CLC polynomial d-homogeneous $p \in \mathbb{R}_+[\boldsymbol{x}]$

- **Definition:** $\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} p$ is log-concave for all $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}_+^n$.
- **Definition:** Matroidal support + all derivatives $\partial_{x_1}^{\mu_1} \cdots \partial_{x_n}^{\mu_n} p$ with $|\boldsymbol{\mu}| = d - 2$ are quadratic forms with Lorentz signature.
- **Intuition:** Generalizes real stability to further capture log-concavity: for $n = 2$, Lorentzian/CLC is equivalent to ULC coefficients.
- **Intuition:** Lorentz signature is equivalent to a **reverse Cauchy-Schwarz** inequality or **Alexandrov-Fenchel** inequality.

Preservers via [Brändén-Huh], [Anari-Liu-Oveis Gharan-Vinzant]

- Same method for determining if a linear operator preserves Lorentzian.
- Unfortunately **not** a characterization.
- The [Symb[T] is Lorentzian $\implies$ T preserves Lorentzian] direction still holds. (The practical direction.)
- **E.g.:** $\nabla_{\boldsymbol{v}} p$ for $\boldsymbol{v} \in \mathbb{R}_+^n$, $p(A\boldsymbol{x})$ for A with ≥ 0 entries

So far in the course: Polynomial capacity

Recall: Given polynomial p with coefficients ≥ 0 and any $\boldsymbol{\alpha} \in \mathbb{R}_+^n$, define

$$\mathrm{Cap}_{\boldsymbol{\alpha}}(p) := \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\boldsymbol{\alpha}}} = \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{x})}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}.$$

Some basic facts:

- $\mathrm{Cap}_{\boldsymbol{\alpha}}(p) > 0$ iff $\boldsymbol{\alpha} \in \mathrm{Newt}(p)$.
- $\mathrm{Cap}_{\boldsymbol{\alpha}}(p) = p(\mathbf{1})$ iff $\boldsymbol{\alpha} = \nabla \log p(\mathbf{1})$.
- $\mathrm{Cap}_{\boldsymbol{\mu}}(p) \geq p_{\boldsymbol{\mu}}$ for $\boldsymbol{\mu} \in \mathbb{Z}_+^n$.

Gurvits' theorem: For n-homogeneous real stable $p \in \mathbb{R}_+[x_1, \ldots, x_n]$,

$$\mathrm{Cap}_{\mathbf{1}}\left(\partial_{x_n}\big|_{x_n=0} \, p \right) \geq \left(\frac{n-1}{n} \right)^{n-1} \mathrm{Cap}_{\mathbf{1}}(p).$$

Gurvits' corollary: $\mathrm{Cap}_{\mathbf{1}}(p) \geq p_{\mathbf{1}} \geq \dfrac{n!}{n^n} \mathrm{Cap}_{\mathbf{1}}(p)$.

Implies e^n-approximation algorithm to the permanent, and other things...

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What/why/how: Coefficient bounds via capacity

Recall: Given polynomial p with coefficients ≥ 0 and any $\boldsymbol{\alpha} \in \mathbb{R}_+^n$, define

$$\mathrm{Cap}_{\boldsymbol{\alpha}}(p) := \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\boldsymbol{\alpha}}} = \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{x})}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}.$$

Want: Given polynomial $p(\boldsymbol{x}) = \sum_{\boldsymbol{\mu}} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$, obtain bound of the form

$$\mathrm{Cap}_{\boldsymbol{\mu}}(p) \geq p_{\boldsymbol{\mu}} \geq K(\mu_1, \ldots, \mu_n) \cdot \mathrm{Cap}_{\boldsymbol{\mu}}(p).$$

Why we care: Combinatorial bounds when $\mathrm{Cap}_{\boldsymbol{\mu}}(p)$ has explicit formula, or else algorithmic bounds since $\mathrm{Cap}_{\boldsymbol{\mu}}(p)$ is essentially a convex program.

How do we get such bounds? Upper bound easy; lower bound:

1. Obtain capacity bounds on coefficients of univariate (or bivariate homogeneous) polynomials.
2. Apply such bounds to $p(y_1, \ldots, y_{n-1}, t) \in \mathbb{R}_+[t]$ (or $p(y_1 \cdot s, \ldots, y_{n-1} \cdot s, t) \in \mathbb{R}_+[t, s]$) for any fixed $y_1, \ldots, y_{n-1} > 0$.
3. Take inf over $y_1, \ldots, y_{n-1}$ and induct.

Applications we have seen

Capacity bounds for **real stable**, **Lorentzian**, **denormalized Lorentzian**.

Permanent (Gurvits): Given matrix A, define $p(\boldsymbol{x}) := \prod_{i=1}^n \sum_{j=1}^n a_{ij} x_j$:

$$\mathrm{Cap}_{\mathbf{1}}(p) \geq \mathrm{per}(A) = p_{\mathbf{1}} \geq \frac{n!}{n^n} \mathrm{Cap}_{\mathbf{1}}(p) \geq e^{-n} \mathrm{Cap}_{\mathbf{1}}(p).$$

When A is DS (doubly stochastic), we have $\mathrm{Cap}_{\mathbf{1}}(p) = 1$.

Mixed volume (Gurvits): Given convex compact set $K_1, \ldots, K_n \subset \mathbb{R}^n$, consider the polynomial $p(\boldsymbol{x}) := \mathrm{vol}(\sum_{i=1}^n x_i K_i)$ via Minkowski sum:

$$\frac{1}{n!} \mathrm{Cap}_{\mathbf{1}}(p) \geq V(K_1, K_2, \ldots, K_n) = \frac{1}{n!} p_{\mathbf{1}} \geq \frac{1}{n^n} \mathrm{Cap}_{\mathbf{1}}(p).$$

When $(K_1, \ldots, K_n)$ is a "DS tuple", we have $\mathrm{Cap}_{\mathbf{1}}(p) = 1$.

Similar bounds for $V(K_1^{\mu_1}, \ldots, K_n^{\mu_d})$ when $K_i \subset \mathbb{R}^d$ and $|\boldsymbol{\mu}| = d$ in terms of $\mathrm{Cap}_{\boldsymbol{\mu}}(p)$, where $K_i^{\mu_i}$ indicates μ_i copies of K_i.

Also: **perfect matchings**, **mixed discriminant**, **contingency tables**

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Univariate bounds

Recall: How do we get such bounds?

1. **Obtain capacity bounds on coefficients of univariate (or bivariate homogeneous) polynomials.**
2. Apply such bounds to $p(y_1, \ldots, y_{n-1}, t) \in \mathbb{R}_+[t]$ (or $p(y_1 \cdot s, \ldots, y_{n-1} \cdot s, t) \in \mathbb{R}_+[t, s]$) for any fixed $y_1, \ldots, y_{n-1} > 0$.
3. Take inf over $y_1, \ldots, y_{n-1}$ and induct.

First: How are the univariate and bivariate homogeneous cases related?
Capacity relation for $p(t) = P(t, 1)$ where P is homogeneous:

$$\mathrm{Cap}_k(p) = \inf_{t>0} \frac{\sum_{i=0}^d p_i t^i}{t^k} = \inf_{t,s>0} \frac{\sum_{i=0}^d p_i \left(\frac{t}{s}\right)^i \cdot s^d}{\left(\frac{t}{s}\right)^k \cdot s^d} = \mathrm{Cap}_{(k,d-k)}(P).$$

Now: How do we obtain univariate bounds?

Univariate bounds

Lemma (Brändén-L-Pak '20)

Let $q, w \in \mathbb{R}_+^d[t]$ be such that $\left(\frac{q_j}{w_j}\right)_{j=0}^d$ forms a log-concave sequence. For all $k \in \{0, \ldots, d\}$, we have

$$q_k \geq \frac{w_k}{\mathrm{Cap}_k(w)} \cdot \mathrm{Cap}_k(q).$$

Proof sketch:

1. WLOG $q_k = w_k = 1$ by scaling: now want $\mathrm{Cap}_k(q) \leq \mathrm{Cap}_k(w)$.
2. Log-concavity implies $\frac{q_{k+j}}{w_{k+j}} \leq \left(\frac{q_{k+1}}{w_{k+1}}\right)^j$ for all j (since $\frac{q_k}{w_k} = 1$).
3. $\frac{q(t)}{t^k} = \sum_{j=-k}^{d-k} q_{k+j} t^j \leq \sum_{j=-k}^{d-k} w_{k+j} \left(\frac{q_{k+1}}{w_{k+1}} \cdot t\right)^j = \frac{w\left(\frac{q_{k+1}}{w_{k+1}} \cdot t\right)}{\left(\frac{q_{k+1}}{w_{k+1}} \cdot t\right)^k}.$
4. Since $\frac{q_{k+1}}{w_{k+1}} > 0$ is fixed, take inf over $t > 0$ to get the result.

Univariate bounds for real-rooted/Lorentzian polynomials

Previous slide: $q_k \geq \frac{w_k}{\text{Cap}_k(w)} \cdot \text{Cap}_k(q)$ whenever $\frac{q_j}{w_j}$ log-concave.

Recall: Real-rooted $\implies$ ULC (ultra log-concave) coefficients.

For bivariate homogeneous: Lorentzian $\iff$ ULC coefficients.

ULC coefficients: $\frac{q_j}{\binom{d}{j}}$ is log-concave for $q \in \mathbb{R}_+^d[t] \implies w_j = \binom{d}{j}$.

Corollary

If $q(t) \in \mathbb{R}_+^d[t]$ has ULC coefficients, then for all k we have

$$q_k \geq \binom{d}{k} \frac{k^k (d-k)^{d-k}}{d^d} \text{Cap}_k(q).$$

Proof: By calculus, $\text{Cap}_k(w) = \inf_{t>0} \frac{(t+1)^d}{t^k} = \frac{d^d}{k^k (d-k)^{d-k}}$.

Multivariate bounds for real stable polynomials

Want: Bound on coefficient $p_{\boldsymbol{\mu}}$ for some $\boldsymbol{\mu} \in \mathbb{Z}_+^n$.

Given real stable $p \in \mathbb{R}_+^{\boldsymbol{\lambda}}[x_1, \ldots, x_n]$, we have that

$$q(t) := p(y_1, \ldots, y_{n-1}, t) \in \mathbb{R}_+^{\lambda_n}[t]$$

is real-rooted for all $y_1, \ldots, y_{n-1} > 0 \implies$ ULC coeffcients.

Previous bound: $q_{\mu_n} \geq \binom{\lambda_n}{\mu_n} \frac{\mu_n^{\mu_n}(\lambda_n - \mu_n)^{\lambda_n - \mu_n}}{\lambda_n^{\lambda_n}} \operatorname{Cap}_{\mu_n}(q)$.

Next: $q_{\mu_n} = \frac{1}{\mu_n!} \cdot \left[\partial_{x_n}^{\mu_n}|_{x_n=0}\, p \right](\boldsymbol{y})$ and

$$\operatorname{Cap}_{\mu_n}(q) = \inf_{t>0} \frac{p(\boldsymbol{y}, t)}{t^{\mu_n}} = \inf_{x_n>0} \frac{p(\boldsymbol{y}, x_n)}{x_n^{\mu_n}}.$$

Since $q_{\mu_n} = \frac{1}{\mu_n!} \cdot \left[\partial_{x_n}^{\mu_n}|_{x_n=0}\, p \right](\boldsymbol{y})$ is real stable as a function of $\boldsymbol{y}$, we can induct by dividing through by $y_1^{\mu_1} \cdots y_{n-1}^{\mu_{n-1}}$ and then take inf over $\boldsymbol{y} > 0$.

Putting it all together

Last slide: For $K_d(k) := \binom{d}{k} \frac{k^k (d-k)^{d-k}}{d^d}$, we have

$$\frac{1}{\mu_n!} \cdot \partial_{x_n}^{\mu_n}\big|_{x_n=0}\, p(\boldsymbol{y}) = q_{\mu_n} \geq K_{\lambda_n}(\mu_n) \cdot \mathrm{Cap}_{\mu_n}(q)$$
$$= K_{\lambda_n}(\mu_n) \cdot \inf_{x_n > 0} \frac{p(\boldsymbol{y}, x_n)}{x_n^{\mu_n}}.$$

Now: Divide through by $y_1^{\mu_1} \cdots y_{n-1}^{\mu_{n-1}}$ and take inf to get

$$\frac{1}{\mu_n!} \cdot \inf_{\boldsymbol{y}>0} \frac{\partial_{x_n}^{\mu_n}\big|_{x_n=0}\, p(\boldsymbol{y})}{y_1^{\mu_1} \cdots y_{n-1}^{\mu_{n-1}}} \geq K_{\lambda_n}(\mu_n) \cdot \inf_{\boldsymbol{y}, x_n > 0} \frac{p(\boldsymbol{y}, x_n)}{y_1^{\mu_1} \cdots y_{n-1}^{\mu_{n-1}} x_n^{\mu_n}}.$$

Theorem (Gurvits)

Given a real stable $p \in \mathbb{R}_+^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ and $\boldsymbol{\mu} \in \mathbb{Z}_+^n$, we have

$$\mathrm{Cap}_{(\mu_1,\ldots,\mu_{n-1})}\left(\frac{1}{\mu_n!} \cdot \partial_{x_n}^{\mu_n}\big|_{x_n=0}\, p\right) \geq \binom{\lambda_n}{\mu_n} \frac{\mu_n^{\mu_n}(\lambda_n - \mu_n)^{\lambda_n - \mu_n}}{\lambda_n^{\lambda_n}} \mathrm{Cap}_{\boldsymbol{\mu}}(p).$$

Coefficient bounds for real stable polynomials

Next: Use induction to obtain a general coefficient bound.

Corollary (Gurvits)

Given a real stable $p \in \mathbb{R}_+^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ and $\boldsymbol{\mu} \in \mathbb{Z}_+^n$, we have

$$p_{\boldsymbol{\mu}} \geq \left[\prod_{i=1}^n \binom{\lambda_i}{\mu_i} \frac{\mu_i^{\mu_i}(\lambda_i - \mu_i)^{\lambda_i - \mu_i}}{\lambda_i^{\lambda_i}}\right] \cdot \mathrm{Cap}_{\boldsymbol{\mu}}(p).$$

Base case: Univariate case: $p_k \geq \binom{d}{k} \frac{k^k (d-k)^{d-k}}{d^d} \mathrm{Cap}_k(p)$.

Induction: Apply bound to $q := \frac{1}{\mu_n!} \cdot \partial_{x_n}^{\mu_n}|_{x_n=0}\, p$ and $\boldsymbol{\nu} := (\mu_1, \ldots, \mu_{n-1})$:

$$p_{\boldsymbol{\mu}} = q_{\boldsymbol{\nu}} \geq \left[\prod_{i=1}^{n-1} \binom{\lambda_i}{\mu_i} \frac{\mu_i^{\mu_i}(\lambda_i - \mu_i)^{\lambda_i - \mu_i}}{\lambda_i^{\lambda_i}}\right] \cdot \mathrm{Cap}_{\boldsymbol{\nu}}(q).$$

Now apply theorem from previous slide and combine:

$$\mathrm{Cap}_{\boldsymbol{\nu}}(q) = \mathrm{Cap}_{\boldsymbol{\nu}}\left(\frac{1}{\mu!} \cdot \partial_{x_n}^{\mu_n}|_{x_n=0}\, p\right) \geq \binom{\lambda_n}{\mu_n} \frac{\mu_n^{\mu_n}(\lambda_n - \mu_n)^{\lambda_n - \mu_n}}{\lambda_n^{\lambda_n}} \mathrm{Cap}_{\boldsymbol{\mu}}(p).$$

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Contingency tables

Given vectors $\boldsymbol{\alpha} \in \mathbb{Z}_+^m$ and $\boldsymbol{\beta} \in \mathbb{Z}_+^n$, a **contingency table** is a $m \times n$ matrix $M = (m_{ij})$ with $\mathbb{Z}_+$ entries such that

$$\sum_{i=1}^{m} m_{ij} = \beta_j \quad \text{for all } j \quad \text{and} \quad \sum_{j=1}^{n} m_{ij} = \alpha_i \quad \text{for all } i.$$

Definition: $\text{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \#$ of contingency tables with "marginals" $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

E.g.: For $n = m$, $\text{CT}(d \cdot \mathbf{1}, d \cdot \mathbf{1})$ is the number of (non-simple) d-regular bipartite graphs on $2n$ vertices. (Similar interpretation more generally.)

E.g.: 2×3 table with marginals $\boldsymbol{\alpha} = (1, 4)$ and $\boldsymbol{\beta} = (2, 2, 1)$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

Definition: $\text{BCT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \#$ of contingency tables with "marginals" $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and all entries either 0 or 1. **E.g.:** For $n = m$, $\text{BCT}(\mathbf{1}, \mathbf{1}) = n!$.

Contingency tables generating function

Contingency tables generating function:

$$f(\boldsymbol{x}, \boldsymbol{y}) := \prod_{i=1}^{m} \prod_{j=1}^{n} \left(1 + x_i y_j + (x_i y_j)^2 + \cdots\right) = \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cdot \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{\boldsymbol{\beta}}.$$

Why? Contingency table given by M: $m_{ij} = k \iff (x_i y_j)^k$.

Binary contingency tables generating function:

$$p(\boldsymbol{x}, \boldsymbol{y}) := \prod_{i=1}^{m} \prod_{j=1}^{n} (1 + x_i y_j) = \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathrm{BCT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cdot \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{\boldsymbol{\beta}}.$$

Now define $\boldsymbol{\gamma} := (m, m, \ldots, m)$ and consider:

$$\tilde{p}(\boldsymbol{x}, \boldsymbol{y}) := \prod_{i=1}^{m} \prod_{j=1}^{n} (y_j + x_i) = \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathrm{BCT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cdot \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{\boldsymbol{\gamma} - \boldsymbol{\beta}}.$$

Nice: Real stable polynomial with coefficients which count BCT.

Capacity bounds for binary contingency tables

Last slide: For $\boldsymbol{\lambda} := (n, \ldots, n)$ and $\boldsymbol{\gamma} := (m, \ldots, m)$, consider:

$$\tilde{p}(\boldsymbol{x}, \boldsymbol{y}) := \prod_{i=1}^{m} \prod_{j=1}^{n} (y_j + x_i) = \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathrm{BCT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cdot \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{\boldsymbol{\gamma} - \boldsymbol{\beta}} \in \mathbb{R}_+^{(\boldsymbol{\lambda}, \boldsymbol{\gamma})}[\boldsymbol{x}, \boldsymbol{y}].$$

Recall: For real stable $p \in \mathbb{R}_+^{\boldsymbol{\lambda}}[\boldsymbol{x}]$ and $\boldsymbol{\mu} \in \mathbb{Z}_+^n$, we have

$$p_{\boldsymbol{\mu}} \geq \prod_{i=1}^{n} \binom{\lambda_i}{\mu_i} \frac{\mu_i^{\mu_i} (\lambda_i - \mu_i)^{\lambda_i - \mu_i}}{\lambda_i^{\lambda_i}} \mathrm{Cap}_{\boldsymbol{\mu}}(p).$$

Therefore: $\mathrm{BCT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is bounded below by

$$\prod_{i=1}^{m} \binom{n}{\alpha_i} \frac{\alpha_i^{\alpha_i} (n - \alpha_i)^{n - \alpha_i}}{n^n} \prod_{j=1}^{n} \binom{m}{\beta_j} \frac{\beta_j^{\beta_j} (m - \beta_j)^{m - \beta_j}}{m^m} \mathrm{Cap}_{(\boldsymbol{\alpha}, \boldsymbol{\gamma} - \boldsymbol{\beta})}(\tilde{p}).$$

Sanity check: Counting permutations

Let's try $n = m$ and $\boldsymbol{\alpha} = \boldsymbol{\beta} = 1$ (permutations):

$$\mathrm{BCT}(\mathbf{1}, \mathbf{1}) \geq \prod_{i=1}^{n} \left(n \cdot \frac{(n-1)^{n-1}}{n^n} \right) \prod_{j=1}^{n} \left(n \cdot \frac{(n-1)^{n-1}}{n^n} \right) \mathrm{Cap}_{(\mathbf{1}, \boldsymbol{n}-\mathbf{1})}(\tilde{p}).$$

How to compute capacity? One option is to bound $\mathrm{Cap}_{(\mathbf{1},\mathbf{1})}(\tilde{p})$ via:

$$\inf_{\boldsymbol{x},\boldsymbol{y}>0} \frac{\prod_{i=1}^{n} \prod_{j=1}^{n} (y_j + x_i)}{\boldsymbol{x}^{\mathbf{1}} \boldsymbol{y}^{\boldsymbol{n}-\mathbf{1}}} \geq \prod_{i,j=1}^{n} \inf_{x_i, y_j > 0} \left(\frac{y_j + x_i}{x_i^{\frac{1}{n}} y_j^{1-\frac{1}{n}}} \right) \geq n^n \left(\frac{n}{n-1} \right)^{n(n-1)}.$$

Put it all together:

$$\mathrm{BCT}(\mathbf{1}, \mathbf{1}) \geq \left(\frac{n-1}{n} \right)^{2n(n-1)} n^n \left(\frac{n}{n-1} \right)^{n(n-1)} = n^n \left(\frac{n-1}{n} \right)^{n(n-1)}$$
$$\approx \frac{n!}{\sqrt{2\pi n}} e^n \cdot e^{-(n-1)} = n! \cdot \frac{e}{\sqrt{2\pi n}}.$$

Decent approximation: Off by a factor of $\sqrt{n}$.

General contingency tables

Recall: Contingency tables generating function:

$$f(\boldsymbol{x}, \boldsymbol{y}) := \prod_{i=1}^{m} \prod_{j=1}^{n} \left(1 + x_i y_j + (x_i y_j)^2 + \cdots \right) = \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathsf{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cdot \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{\boldsymbol{\beta}}.$$

Actually: We can cut off the series at $d := \max\{\alpha_i, \beta_j\}$. Same as before:

$$\tilde{f}_d(\boldsymbol{x}, \boldsymbol{y}) := \prod_{i=1}^{m} \prod_{j=1}^{n} \left(y_j^d + x_i y_j^{d-1} + \cdots + x_i^d\right) \cong \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta} \leq \boldsymbol{d}} \mathsf{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cdot \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{\boldsymbol{d} - \boldsymbol{\beta}}.$$

Problem: What class does the polynomial $\sum_{k=0}^{d} x_i^k y_j^{d-k}$ fit into?

Answer: Class of **denormalized Lorentzian** polynomials. Bivariate homogeneous equivalent to log-concave coefficients.

Bonus: We are counting lattice points in various polytopes. By scaling and limiting, we can achieve lower bounds on volumes of these polytopes.

E.g.: Birkhoff polytope, flow polytopes, transportation polytopes