Coefficient Bounds Exercises

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Note: Exercises with a $*$ next to them are more challenging.

Definition. Given a polynomial $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ and $\boldsymbol{\alpha} \in \mathbb{R}_+^n$, we define

$$\operatorname{Cap}_{\boldsymbol{\alpha}}(p) := \inf_{\boldsymbol{x} > 0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\boldsymbol{\alpha}}}.$$

Definition. A homogeneous polynomial $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ is said to be **denormalized Lorentzian** if the polynomial

$$N(p) := \sum_{\boldsymbol{\mu} \in \operatorname{supp}(p)} p_{\boldsymbol{\mu}} \frac{x^{\boldsymbol{\mu}}}{\mu_1! \cdots \mu_n!}$$

is Lorentzian. The operator N is called the **normalization operator**.

Lemma (Closure properties, denormalized Lorentzian)**.** *The set of denormalized Lorentzian polynomials is closed under multiplication, and under the operation $p(x_1, x_2, x_3, \ldots) \mapsto p(x_1, x_1, x_3, \ldots)$.*

Lemma (Univariate capacity bound)**.** *Given $p, w \in \mathbb{R}_+^d[t]$ such that $\left(\frac{p_j}{w_j}\right)_{j=0}^d$ forms a log-concave sequence, we have for all $k \in \{0, 1, \ldots, d\}$ that*

$$p_k \geq \frac{w_k}{\operatorname{Cap}_k(w)} \cdot \operatorname{Cap}_k(p).$$

Exercises

1. Using the results and techniques from this week's lecture, prove the following coefficient bound for a d-homogeneous Lorentzian polynomial $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ and any $\boldsymbol{\mu} \in \mathbb{Z}_+^n$:

   $$p_{\boldsymbol{\mu}} \geq \binom{d}{\boldsymbol{\mu}} \frac{\mu_1^{\mu_1} \cdots \mu_n^{\mu_n}}{d^d} \operatorname{Cap}_{\boldsymbol{\mu}}(p).$$

   Here, $\binom{d}{\boldsymbol{\mu}} = \frac{d!}{\mu_1! \cdots \mu_n!}$ denotes the mutlinomial coefficient.

2. Let $p(\boldsymbol{x}) := \left(\sum_{i=1}^n c_i x_i\right)^d$ be a power of a linear form with $c_i \geq 0$ for all i. Prove the following formula for the capacity of p given any $\boldsymbol{\alpha} \in \mathbb{R}_+^n$:

   $$\operatorname{Cap}_{\boldsymbol{\alpha}}(p) = \prod_{i=1}^n \left(\frac{d \cdot c_i}{\alpha_i}\right)^{\alpha_i}.$$

   Note: This bounds holds for **all** $\boldsymbol{\alpha} \in \mathbb{R}_+^n$, not just integer vectors.

3. Prove that for all bivariate d-homogeneous denormalized Lorentzian polynomials $p \in \mathbb{R}_+[x, y]$ of the form $\sum_{j=0}^d p_j x^j y^{d-j}$, we have for all $k \in \{0, 1, \ldots, d\}$ that

   $$p_k \geq \max\left\{\frac{k^k}{(k+1)^{k+1}}, \frac{(d-k)^{d-k}}{(d-k+1)^{d-k+1}}\right\} \cdot \operatorname{Cap}_{(k,d-k)}(p).$$

4. Prove that if the polynomial $x_1^k \cdot p(\boldsymbol{x})$ is denormalized Lorentzian, then so is the polynomial $p(\boldsymbol{x})$. Prove that this is **not** true for Lorentzian polynomials.

5. Taking for granted the closure properties lemma above, prove the following coefficient bound for denormalized Lorentzian polynomials. Let $p \in \mathbb{R}_+^{\boldsymbol{\lambda}}[x_1, \ldots, x_n]$ be a homogeneous denormalized Lorentzian polynomial of degree λ_i in the variable x_i for all i. Then for any $\boldsymbol{\mu} \in \mathbb{Z}_+^n$ we have

$$p_{\boldsymbol{\mu}} \geq \left[\prod_{i=2}^{n} \max\left\{\frac{\mu_i^{\mu_i}}{(\mu_1+1)^{\mu_i+1}}, \frac{(\lambda_i-\mu_i)^{\lambda_i-\mu_i}}{(\lambda_i-\mu_i+1)^{\lambda_i-\mu_i+1}}\right\}\right] \cdot \mathrm{Cap}_{\boldsymbol{\mu}}(p).$$

   Note that the product starting at $i = 2$ here is **not** a typo. **Hint:** The fact that we can refer to the per-variable degree here relies on Exercise 4. This is why the argument you use to prove this exercise will not quite work for Lorentzian polynomials.

6. Using the previous exercise, prove the following simpler bound for any denormalized Lorentzian polynomial $p \in \mathbb{R}_+^{\boldsymbol{\lambda}}[\boldsymbol{x}]$:

$$p_{\boldsymbol{\mu}} \geq \left[\frac{1}{e^{n-1}} \prod_{i=2}^{n} \frac{1}{\mu_i+1}\right] \cdot \mathrm{Cap}_{\boldsymbol{\mu}}(p).$$

7. $*$ Recall the contingency tables generating function from this week's lecture. Use the previous two exercises and the closure properties for denormalized Lorentzian polynomials to prove the following bound on the number of $m \times n$ contingency tables, for $\boldsymbol{\alpha} \in \mathbb{Z}_+^m$ and $\boldsymbol{\beta} \in \mathbb{Z}_+^n$, where f is the contingency tables generating function:

$$\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \geq \left[\frac{1}{e^{m+n-1}} \prod_{i=2}^{m} \frac{1}{\alpha_i+1} \prod_{j=1}^{n} \frac{1}{\beta_j+1}\right] \cdot \mathrm{Cap}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(f).$$

   Hint: The generating function is a power series, and so one needs to truncate the series to be able to apply the above results. Also one needs to "flip the degree" of the truncations, like we had to do in the BCT case in the lecture. All that said, this result is not quite as straightforward from Exercise 6 as it may appear.

8. $*$ Use the previous exercise and ideas from Exercise 2 to obtain an explicit lower bound on the volume of the Birkhoff polytope, which is defined to be the set of $n \times n$ matrices with non-negative entries whose rows and columns all sum to 1.