Capacity Recap: All the forms of capacity

Polynomial Capacity: Theory, Applications, Generalizations

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Outline

1. Last time: Invariant-theoretic capacity
   - Null-cone problem
   - Invariant-theoretic capacity and the scaling-type algorithm
   - Scaling for real stable polynomials
2. Remembering all the forms of capacity
   - Polynomial capacity
   - Entropic capacity
   - Matrix capacity
   - Invariant-theoretic (non-commutative) capacity
3. Thanks!

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The null-cone problem

Let $\pi : G \to \mathrm{GL}(V)$ be a representation of a group G (i.e., π is a group homomorphism and V is a vector space).

Definition: An **orbit** of $v \in V$ is the set $\mathcal{O}_v := \{\pi(g)v : g \in G\} \subset V$.

Definition: The **null-cone** of V or π is the set $\{v : 0 \in \overline{\mathcal{O}_v}\}$.

[Hilbert], [Mumford '65]: v is in the null-cone iff for every non-constant homogeneous G-invariant polynomial p on V we have $p(v) = 0$.

E.g.: v in null-cone $\implies \pi(g_i)v \to 0 \implies p(v) = p(\pi(g_i)v) = p(0) = 0$.

[Kempf-Ness '79]: v is *not* in the null-cone iff $\mu(w) = 0$ for some $w \in \overline{\mathcal{O}_v}$, where μ is the **moment map** of π.

Moment map: Something like the "gradient" of the action of π at $g = \mathrm{id}$:

$$\text{“}\mu(w) = \nabla|_{X=0} \log \|\pi(e^X)w\|\text{”}.$$

Convex programming: $f = \|w\|$ attains minimum at w_0 iff $\nabla f(w_0) = 0$.

The moment map and moment polytope

Throughout: Think $G = \mathrm{GL}_n(\mathbb{C})$ or $G = \mathbb{T}^n$ with $\pi : G \to \mathrm{GL}(V)$.

Definition: The moment map $\mu(v)$ for $v \in V$ is defined via

$$\langle H, \mu(v) \rangle := \partial_t|_{t=0} \log \|\pi(e^{tH})v\|,$$

and $\mu(v)$ is Hermitian for $\mathrm{GL}_n(\mathbb{C})$ or a real (diagonal) vector for $\mathbb{T}^n$.

Idea: $\mu(v)$ is the "gradient" of $\log \|\pi(e^X)v\|$ at $X = 0$.

Moment polytope: $\Delta(v) := \overline{\{\mathrm{eig}(\mu(w)) : w \in \mathcal{O}_v\}}$ is a convex polytope.

Kempf-Ness: v *not* in null-cone iff $\mu(w) = 0$ for a $w \in \overline{\mathcal{O}_v}$ **iff** $\mathbf{0} \in \Delta(v)$.

Recall: In the commutative case $(G = \mathbb{T}^n)$, we have:

- $\inf_{g \in G} \|\pi(g) \cdot v\|_2^2 = \inf_{\boldsymbol{x} > 0} \sum_k |c_k|^2 \boldsymbol{x}^{2\omega_k}$ is a capacity problem.
- **Kempf-Ness:** $\mathrm{Cap}_{\mathbf{0}}(\sum_k |c_k|^2 \boldsymbol{x}^{2\omega_k}) > 0$ iff $\mathbf{0} \in \mathrm{Newt}(\sum_k |c_k|^2 \boldsymbol{x}^{2\omega_k})$.
- Roughly equivalent to polynomial capacity.

Invariant-theoretic capacity

Last slide: $\inf_g \|\pi(g)v\|$ is a capacity problem in the commutative case.

In more general cases, let's just **make this the definition**:

$$\mathrm{Cap}_{\mathbf{0}}(v) := \inf_{g \in G} \|\pi(g)v\|.$$

"Non-commutative" capacity, "invariant-theoretic" capacity, etc.

Also called **non-commutative geometric programming** since the commutative case captures unconstrained **geometric programming** (see [Bürgisser-Li-Nieuwboer-Walter '20]).

Kempf-Ness: $\mathrm{Cap}_{\mathbf{0}}(v) > 0$ iff $\mathbf{0}$ is in the moment polytope $\implies$ Generalization of the same statement for polynomial capacity.

Recall: $\inf_{\boldsymbol{y} \in \mathbb{R}^n} \log \sum_{k=1}^n |c_k|^2 e^{\langle \boldsymbol{y}, 2\omega_k \rangle}$ is a convex program. **Can we do the same thing to non-commutative capacity?**

Appears to be "no"... but general capacity is still **geodesically convex**. **And, there is a scaling-type algorithm.**

Scaling-type algorithm

Recall the scaling-type algo, where the measure of progress is $\mu = \text{Cap}_{\mathbf{0}}$:

1. **"Preprocess":** Set $g_0 = \text{id}$.
2. **Iterations:** Geodesic gradient descent, Taylor approx, "trust-region" methods... **I.e.:** Natural analogs to convex Euclidean techniques.
3. **Approximation:** How close do we need to get before stopping?

Approximation step is key to determine computational complexity.

Theorem [BFGOWW '19]: For $\|v\| = 1$, we have

$$1 - \frac{\|\mu(v)\|}{\gamma(\pi)} \leq [\text{Cap}_{\mathbf{0}}(v)]^2 \leq 1 - \frac{\|\mu(v)\|^2}{4N(\pi)^2}.$$

Corollary: $\mathbf{0} \in \Delta(v)$ iff $\Delta(v)$ contains a point smaller than $\gamma(\pi)$.
(This $\gamma(\pi)$ *is* how close we must get before stopping.)

Definition: The **weight margin** $\gamma(\pi)$ is the minimum distance between $\mathbf{0}$ and any subset of the "weights" whose convex hull does not contain $\mathbf{0}$.

Weight margin examples

Last slide: The **weight margin** $\gamma(\pi)$ is the minimum distance between $\mathbf{0}$ and any subset of the "weights" whose convex hull does not contain $\mathbf{0}$.

Matrix scaling: Action of $(\mathrm{S}\mathbb{T}^n)^2$ via left-right action on matrices. $\gamma(\pi) \geq \frac{1}{\mathrm{poly}(n)}$ via [Linial-Samorodnitsky-Wigderson '00].

Operator scaling: Action of $(\mathrm{SL}_n(\mathbb{C}))^2$ on $(M_1, \ldots, M_\ell)$ via simultaneous left-right action. $\gamma(\pi) \geq \frac{1}{\mathrm{poly}(n)}$ via [Gurvits '04], [GGOW '15].

Tensor scaling for 3-tensors: Action of $(\mathrm{GL}_n(\mathbb{C}))^3$ on 3-tensors. $\gamma(\pi) \leq 2^{-\mathrm{poly}(n)}$ via [Franks-Reichenbach '21] (the other day).

Last result: Negative result for this method. **Open:** Other methods?

Real stable polynomials formulation: Given a real stable polynomial with $\mathbf{1}$ not in its Newton polytope, how far away can Newton polytope be?

Weight margin for real stable polynomials

Theorem [BFGOWW '19]: For $\|v\| = 1$, we have

$$\text{Cap}_{\mathbf{0}}(v) = \inf_{g \in G} \|\pi(g)v\|_2 \geq \left(1 - \frac{\|\mu(v)\|}{\gamma(\pi)}\right)^{1/2}.$$

Theorem [L-Gurvits '20]: Fix an n-homogeneous n-variate real stable p with non-negative coefficients. If $p(\mathbf{1}) = 1$ and $\|\mathbf{1} - \nabla p(\mathbf{1})\|_1 < 2$, then

$$\text{Cap}_{\mathbf{1}}(p) = \inf_{\boldsymbol{x} > 0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\mathbf{1}}} \geq \left(1 - \frac{\|\mathbf{1} - \nabla p(\mathbf{1})\|_1}{2}\right)^n.$$

Corollary: If $\mathbf{1} \notin \text{Newt}(p)$, then $\|\mathbf{1} - \text{Newt}(p)\|_1 \geq 2$.

Easy: (Kempf-Ness) $\mathbf{1} \notin \text{Newt}(p)$ iff $\text{Cap}_{\mathbf{1}}(p) = 0$.

$\implies$ Above bound cannot hold.

$\implies$ $\|\mathbf{1} - \nabla p(\mathbf{1})\|_1 \geq 2$.

Finally: "Scale" p to have marginals close to the boundary of $\text{Newt}(p)$.

Corollary: Nice weight margin for special subclass of polynomials. **Can we do something similar in the more general non-commutative case?**

Scaling for real stable polynomials

Proof: Take any α in the relative interior of $\mathrm{Newt}(p)$.

- $\mathrm{Cap}_{\alpha}(p) > 0$ **and** there is a $\boldsymbol{y} > 0$ such that $\frac{p(\boldsymbol{y})}{\boldsymbol{y}^{\alpha}} = \mathrm{Cap}_{\alpha}(p)$.
- Defining $q(x) := \frac{p(\boldsymbol{y} \circ \boldsymbol{x})}{p(\boldsymbol{y})}$, we have

$$\mathrm{Cap}_{\alpha}(q) = \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{y} \circ \boldsymbol{x})}{\boldsymbol{x}^{\alpha} \cdot p(\boldsymbol{y})} = \frac{\boldsymbol{y}^{\alpha}}{p(\boldsymbol{y})} \cdot \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{y} \circ \boldsymbol{x})}{(\boldsymbol{y} \circ \boldsymbol{x})^{\alpha}} = \frac{\boldsymbol{y}^{\alpha}}{p(\boldsymbol{y})} \cdot \mathrm{Cap}_{\alpha}(p) = 1$$

  which implies $\nabla q(\mathbf{1}) = \alpha$.
- q is the **scaling** of p which makes the marginals equal to α.
- **Now:** $\mathrm{Newt}(q) = \mathrm{Newt}(p)$ and $\nabla q(\mathbf{1}) = \alpha \implies \|\mathbf{1} - \alpha\|_1 \geq 2$.

Works for any α in relative interior of $\mathrm{Newt}(p)$, so $\|\mathbf{1} - \mathrm{Newt}(p)\|_1 \geq 2$.

Scaling step: Precomposing by $\boldsymbol{y}$ updates "column sums":

$$M \to \prod_{i=1}^{n} \sum_{j=1}^{n} m_{ij} x_j \to \prod_{i=1}^{n} \sum_{j=1}^{n} m_{ij} y_j x_j \to M \cdot \mathrm{diag}(\boldsymbol{y}).$$

Then: Dividing by $p(\boldsymbol{y})$ updates "row sums": $\prod_i \sum_j m_{ij} \cdot 1 = 1$.

An aside: Orbit closure of polynomial scaling

Fix real stable $p(x_1, \ldots, x_n)$ and scale by $\boldsymbol{y} > 0$: $p \to q := \frac{p(\boldsymbol{y} \circ \boldsymbol{x})}{p(\boldsymbol{y})}$.

First: Since $\boldsymbol{y} > 0$, we have that $\operatorname{supp}(q) = \operatorname{supp}(p)$.

Question: What supports can appear in the closure?

Answer: Precisely the collection of faces of $\mathrm{Newt}(p)$ intersected with $\mathbb{Z}^n$.

Proof: First, real stable polynomials have special "convexity property":

- If $\boldsymbol{x} \in \mathbb{Z}^n \cap \mathrm{Newt}(p)$, then $\boldsymbol{x} \in \operatorname{supp}(p)$.

Fix $p(\boldsymbol{x}) = \sum_\mu p_\mu \boldsymbol{x}^\mu \implies$ scaling by $\boldsymbol{y}$ is $q(\boldsymbol{x}) = \frac{\sum_\mu p_\mu \boldsymbol{y}^\mu \boldsymbol{x}^\mu}{\sum_\mu p_\mu \boldsymbol{y}^\mu}$.

Consider $\boldsymbol{y}_t := e^{t\boldsymbol{w}}$ for any $\boldsymbol{w} \in \mathbb{R}^n \implies q_t(\boldsymbol{x}) = \frac{\sum_\mu p_\mu e^{t\langle \boldsymbol{w}, \boldsymbol{\mu} \rangle} \boldsymbol{x}^\mu}{\sum_\mu p_\mu e^{t\langle \boldsymbol{w}, \boldsymbol{\mu} \rangle}}$.

As $t \to \infty$, terms which dominate are those for which $\langle \boldsymbol{w}, \boldsymbol{\mu} \rangle$ is maximized. Corresponds precisely to the faces of $\mathrm{Newt}(p)$ intersected with $\operatorname{supp}(p)$.

Similar situation for torus actions in general (Levent's talk yesterday).

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Polynomial capacity

Given a polynomial $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ and $\boldsymbol{\alpha} \in \mathbb{R}^n_+$, we define

$$\mathrm{Cap}_{\boldsymbol{\alpha}}(p) := \inf_{\boldsymbol{x} > 0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\boldsymbol{\alpha}}} = \inf_{\boldsymbol{y} \in \mathbb{R}^n} \sum_{\boldsymbol{\mu} \in \mathrm{supp}(p)} p_{\boldsymbol{\mu}} e^{\langle \boldsymbol{y}, \boldsymbol{\mu} - \boldsymbol{\alpha} \rangle}.$$

Some basic properties:

- Log-convex program as function of $\boldsymbol{y} \in \mathbb{R}^n$ (geometric programming).
- $\mathrm{Cap}_{\boldsymbol{\alpha}}(p) > 0$ iff $\boldsymbol{\alpha} \in \mathrm{Newt}(p)$.
  - **Kempf-Ness interpretation:** $\boldsymbol{\alpha} \in \mathrm{Newt}(p)$ iff gradient $= \mathbf{0}$ can achieved when optimizing the objective.
  - **Entropy interpretation:** When $p(\mathbf{1}) = 1$, $-\log \mathrm{Cap}_{\boldsymbol{\alpha}}(p)$ is the entropy of a distribution on $\mathrm{supp}(p)$ with expectation $\boldsymbol{\alpha}$.
- If $p(\mathbf{1}) = 1$, then $\mathrm{Cap}_{\boldsymbol{\alpha}}(p) = 1$ iff $\nabla p(\mathbf{1}) = \boldsymbol{\alpha}$.
  - **Entropy interpretation:** $-\log \mathrm{Cap}_{\boldsymbol{\alpha}}(p)$ is the minimum relative entropy of a distribution on $\mathrm{supp}(p)$ with expectation $\boldsymbol{\alpha}$.
  - **Scaling interpretation:** Cap was the measure of progress: once maximized, we are already scaled correctly.

Capacity and real stable / Lorentzian polynomials

Gurvits' theorem: Real stable (or Lorentzian) polynomial p with non-negative coefficients in n variables of homogeneous degree n:

$$p_{\mathbf{1}} \geq \frac{n!}{n^n} \cdot \mathrm{Cap}_{\mathbf{1}} .$$

Corollary: Doubly stochastic $p \implies p_{\mathbf{1}} \geq \frac{n!}{n^n}$ (permanent VdW bound).

[Anari-Oveis Gharan], [Gurvits-L]: Real stable p, q of degree at most λ_k in x_k, and $\boldsymbol{\alpha} \in \mathbb{R}^n_+$:

$$\langle p, q \rangle = \sum_{\boldsymbol{\mu}} \binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}}^{-1} p_{\boldsymbol{\mu}} q_{\boldsymbol{\mu}} \geq K(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \cdot \mathrm{Cap}_{\boldsymbol{\alpha}}(p) \, \mathrm{Cap}_{\boldsymbol{\alpha}}(q).$$

[Anari-Liu-Oveis Gharan-Vinzant]: Similar bound for Lorentzian / CLC.

[Gurvits-L]: T preserves real stability and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n_+$:

$$\frac{\mathrm{Cap}_{\boldsymbol{\beta}}(T[p])}{\mathrm{Cap}_{\boldsymbol{\alpha}}(p)} \geq K(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cdot \mathrm{Cap}_{(\boldsymbol{\beta}, \boldsymbol{\alpha})}(\mathrm{Symb}[T]).$$

Applications of polynomial capacity

Applications include bounds/approximations for:

- Permanent and mixed discriminant (Gurvits): $T(p) = \partial_{x_n}|_{x_n=0}\, p$
- Contingency tables (Barvinok, Barvinok-Hartigan, Gurvits, Brändén-L-Pak) and Eulerian orientations (Csikvári-Schweitzer):

$$T(p) = \partial_{\mathbf{x}}^S p \quad \text{for specially chosen } S$$

- Biregular bipartite k-matchings (Gurvits-L):

$$T(p) = \sum_{S \in \binom{[n]}{k}} \partial_{\mathbf{x}}^S p$$

- Counting/optimization on stable matroids (Straszak-Vishnoi, Anari-Oveis Gharan) and intersection of two general matroids (Anari-Oveis Gharan-Vinzant):

$$T(p) = \sum_{B \in \mathcal{M}} \partial_{\mathbf{x}}^B p$$

Entropic capacity

Dual formulation of polynomial capacity: $p(\mathbf{1}) = 1$, consider a distribution μ on $\mathrm{supp}(p)$ given by $\mu(\boldsymbol{\kappa}) = p_{\boldsymbol{\kappa}}$. Then $-\log \mathrm{Cap}_{\boldsymbol{\alpha}}(p) =$

$$\inf_{\mathbb{E}[\nu]=\boldsymbol{\alpha}} \mathrm{D}_{\mathrm{KL}}(\nu\|\mu) = \inf_{\mathbb{E}[\nu]=\boldsymbol{\alpha}} \sum_{\boldsymbol{\kappa}\in\mathrm{supp}(\mu)} \left[\frac{\nu(\boldsymbol{\kappa})}{\mu(\boldsymbol{\kappa})} \log \frac{\nu(\boldsymbol{\kappa})}{\mu(\boldsymbol{\kappa})}\right] \mu(\boldsymbol{\kappa}).$$

where ν has support contained in $\mathrm{supp}(\mu)$.

Allows for extension to **continuous case:** Fix a measure μ supported on $\Omega \subset \mathbb{R}^n$, and pick some $\boldsymbol{\theta} \in \mathrm{hull}(\Omega)$:

$$\inf_{\substack{\nu=\phi\cdot\mu \\ \mathbb{E}[\nu]=\boldsymbol{\theta}}} \mathrm{D}_{\mathrm{KL}}(\nu\|\mu) = \inf_{\substack{\nu=\phi\cdot\mu \\ \mathbb{E}[\nu]=\boldsymbol{\theta}}} \int_\Omega \phi(\boldsymbol{x}) \log \phi(\boldsymbol{x}) d\mu(\boldsymbol{x}).$$

Define **continuous capacity** via:

$$\mathrm{Cap}_{\boldsymbol{\theta}}(\mu) := \inf_{\boldsymbol{x}>0} \frac{\int_\Omega \boldsymbol{x}^{\boldsymbol{\kappa}} d\mu(\boldsymbol{\kappa})}{\boldsymbol{x}^{\boldsymbol{\theta}}}.$$

As in the discrete case, this is dual to minimum relative entropy.

Maximum entropy distributions

Observation: This shows that **scalings** give rise to maximum entropy (minimum relative entropy) distributions.

Solution given by capacity: If α in the relative interior of Newt(p), then there exists $\boldsymbol{y} > 0$ (the scaling) such that $\nu(\boldsymbol{\kappa}) = \frac{p_{\boldsymbol{\kappa}} \boldsymbol{y}^{\boldsymbol{\kappa}}}{p(\boldsymbol{y})}$. That is,

$$q(\boldsymbol{x}) = \frac{p(\boldsymbol{y} \circ \boldsymbol{x})}{p(\boldsymbol{y})} = \sum_{\boldsymbol{\kappa}} \frac{p_{\boldsymbol{\kappa}} \boldsymbol{y}^{\boldsymbol{\kappa}}}{p(\boldsymbol{y})} \cdot \boldsymbol{x}^{\boldsymbol{\kappa}}$$

is the polynomial whose associated distribution is ν (the KL optimum).

What about continuous case? If $\boldsymbol{\theta}$ in the relative interior of hull(Ω), then there exists $\boldsymbol{y} > 0$ such that $\phi(\boldsymbol{\kappa}) = \frac{\boldsymbol{y}^{\boldsymbol{\kappa}}}{\int_\Omega \boldsymbol{y}^\gamma d\mu(\gamma)}$. That is,

$$q(\boldsymbol{x}) = \frac{p(\boldsymbol{y} \circ \boldsymbol{x})}{p(\boldsymbol{y})} = \int_\Omega \frac{\boldsymbol{y}^{\boldsymbol{\kappa}}}{p(\boldsymbol{y})} \cdot \boldsymbol{x}^{\boldsymbol{\kappa}} d\mu(\boldsymbol{\kappa})$$

is the (torus) scaling of $p(\boldsymbol{x}) = \int_\Omega \boldsymbol{x}^{\boldsymbol{\kappa}} d\mu(\boldsymbol{\kappa})$ with the desired marginals $\boldsymbol{\theta}$.

Matrix capacity

Used as a measure of progress for **operator scaling**:

$$\text{Cap}(T) := \inf_{X \succ 0} \frac{\det(T(X))}{\det(X)},$$

where T is completely positive or at least preserves PSD matrices.

Analog to polynomial case: Can we find positive definite Y such that:

$$\text{Cap}(T) = \frac{\det(T(Y))}{\det(Y)} \implies S(X) := \frac{T(Y^{1/2}XY^{1/2})}{T(Y)},$$

where S is the **scaling** of T to doubly stochastic. **Why?**

$$\text{Cap}(S) = \inf_{X \succ 0} \frac{\det(T(Y^{1/2}XY^{1/2}))}{\det(T(Y))\det(X)} = \inf_{X \succ 0} \frac{\det(T(X))}{\det(T(Y))\det(Y^{-1/2}XY^{-1/2})}$$

This equals $\frac{\det(Y)}{\det(T(Y))} \cdot \text{Cap}(T) = 1 \iff S$ is doubly stochastic.

See also [Franks] for different marginals: $\text{Cap}_A(p) = \inf_{X \succ 0} \frac{\det_A(T(X))}{\det_A(X)}$.

Invariant theoretic (non-commutative) capacity

What we already saw at the beginning (and last week): Let $\pi : G \to \mathrm{GL}(V)$ be a group representation, and define for $v \in V$:

$$\mathrm{Cap}(v) = \mathrm{Cap}_{\mathbf{0}}(v) = \inf_{g \in G} \|\pi(g)v\|_2,$$

the minimum norm over the orbit of v.

In the **commutative (torus) case**, this is precisely a polynomial capacity problem. More generally in this case, **polynomial capacity**, **entropic capacity**, **invariant-theoretic capacity** all coincide. Also **maximum likelihood estimation** (see [Améndola-Kohn-Reichenbach-Seigal]), and **unconstrained geometric programming** (see [Bürgisser-Li-Nieuwboer-Walter]).

In the **left-right action case**, we know that **matrix capacity** and **invariant-theoretic capacity** coincide.

Are there other connections between these notions of capacity?

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Thanks!

Thanks a bunch for joining on this course!

Feel free to ask me about anything capacity-related or stability-related at any time. I am generally interested in any connections to these concepts.

Also feel free to give any feedback on the course. This is only the first of hopefully many times I will discuss this content. And I like to think of myself as someone who can take and utilize criticism, however harsh. So please feel free to test me on this!